THE USE OF STATISTICAL COMMUNICATION THEORY TO CHARACTERIZE POROUS MEDIA

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ABSTRACT

It is known that the static and dynamic behavior of fluids in porous media depends to a large measure on porous-media geometry. In the past, the ability to characterize this geometry has been restricted to such average properties as porosity and permeability. However, in recent years attempts have been made to achieve a more precise characterization based upon the fact that porous media is statistically composed.

In this thesis techniques from statistical communication theory are adapted as possible methods for accomplishing this classification process. In simplest form, a dichotamous function is defined by passing a line through a porous medium, the function having one value when the line is in solid matrix, and another value when the line passes through pore space. The function is then analyzed using (1) Classical Fourier series harmonic analysis, and (2) determination of the autocovariance estimate and power spectrum. Comparisons are made between several functions created from the same medium, and with functions created from other media.

The results indicate that the autocovariance estimate and the power spectrum, as characterizing functions, can

discriminate between different media. This success suggests many more paths of investigation, possibly leading to the complete characterization of porous media through statistical analysis.

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CHAPTER I

INTRODUCTION

In August, 1965, two American astronauts journeyed over two hundred miles into space in preparation for man's future journeys to the moon and thousands of miles beyond. As of August, 1965, man had journeyed only a few hundred feet beneath the surface of the earth upon which he lives. In this very interesting paradox, man is now capable of gathering information many miles into space, but has very little first-hand information about the sub-surface of the very earth from which he launches his space probes. If this seems incongrous, consider that while the benefits of space exploration are speculative at best, man derives the vast majority of his resources, his energy sources, and even his food from this earth. However, there are many explanations for man's actions. Perhaps the foremost is the extreme difficulty encountered in any attempt to explore extensively beneath the earth's surface. Costs would certainly be prohibitive, even if the engineering difficulties could be overcome. Another explanation might be that man has been able to theorize sufficiently about the earth's sub-surface areas from the limited information available so that he has not been willing or seen the necessity of risking great

expense, for what might be very little gain. A less practical explanation, but in all probability a very real reason for the lack of extensive sub-surface exploration, would be that there is less glamour to most people about traveling to the center of the earth than in a journey to Venus or Mars and thus, less interest in making the attempt.

The above discussion is not meant as an implication that man is not interested in the earth's sub-surface. To the contrary, geologists, petroleum engineers, and others who derive their livelihood from the soil and beneath the soil, have performed extensive research in this area. From the exploration of caves and mines, from core samples drawn from wells, and in many other ways, man has been able to learn much about the earth's constitution and composition. As a result of this research, great improvements have been realized in the methods used to extract the natural resources from the earth, not to mention the improved conservation procedures which will allow man to benefit from the earth's resources for many years to come.

As a part of this research, man has spent many years developing intelligence concerning the physical properties of porous media found in the strata of various kinds that lie beneath the earth's surface and the behavior of fluids within these porous media. Research efforts have been concentrated on four properties which are considered fundamental and the basis for discussions regarding all other properties.[1]

These four properties are porosity, a measure of the void space in the media; permeability, a measure of the media's ability to permit flow; the fluid saturation, and the electrical conductivity of the media. We will discuss these properties at greater length in Chapter II.

The basis for this report is derived from a paper by A. E. Scheidegger[2] on "Statistical Geometry of Porous Media," published in 1961. As pointed out by Scheidegger, although the usual geometrical quantities, such as porosity and permeability, give generally good descriptions, there is a need for geometrical description of greater detail. This need arises because of the extreme complexity of the subject matter. For example, in some cases, media with a given porosity will not allow as much flow as other media with less void space because of the lack of continuity in the pore spaces of the first sample. Media with identical permeabilities to one fluid may have different permeabilities to another fluid. These are well-known phenomena in the area of petroleum engineering. As a result, it is sometimes very difficult to classify media of different types without relying on extensive laboratory tests, which are expensive and time consuming. It is the aim of this report, to take advantage of the inherent statistical nature of porous media to characterize the media by a set of numbers or a mathematical function. To clarify the above statement, it is important to point out that porous media are consolidated in

a statistical nature. An individual particle has properties determined by its size, shape, density, electrical conductivity, and various other characteristics. The total media will have properties determined by the properties of the individual particles, and by the means in which the individual particles have united to form the whole. This consolidation process will in turn depend upon the arrangement of the elementary particles, the relative number of different types of particles, and upon the relative number of large and small particles. This dependency upon distribution functions is obviously statistical in nature.

While much time has been devoted to research on the various physical properties mentioned above, very little has been accomplished in the latter area of investigating the statistical nature of porous media. Scheideggar[3] and Flood[4] have provided impetus in this area, and defined some of the goals. It would be particularly desirable, for example, to be able to look at two pieces of porous media through a microscope and tell if they are identical in a statistical sense. It would be even more desirable to be able to make intelligent predictions about the other fundamental properties from the statistical description.

The purpose of this report is to introduce and attempt techniques which utilize statistical communication theory to characterize porous media. In simplest form, the method will consist of defining a function created by passing a line over

or through a porous medium, and assigning a value of +1 when the line passes through pore space, and -1 when it passes through solid matter. This function will then be analyzed in a number of ways. One attempt will be to fit the function with a single Fourier series and determine whether the Fourier coefficients effectively characterize the medium. Brief mention will be made of the double Fourier series whereby a surface area, rather than a single line, might be analyzed.

Finally, the function will be looked at in terms of certain statistical properties, derived from communication theory. The idea of random functions will be introduced and characterizing functions such as the autocovariance function and the power spectrum will be determined. Comparisons will be made within the same medium, and against different media. From the results, it is hoped to ascertain whether a given porous medium can be described and effectively classified by its power spectrum and autocovariance function, to the exclusion of all others.

If any of these three methods can be used to effectively classify porous media, the obvious step to be taken later will be the correlation of these descriptive quantities with the fundamental properties, such as porosity and permeability, which are now used to characterize porous media.

CHAPTER II

HISTORY

Until quite recently, man has investigated and classified porous media largely according to four fundamental properties from which it was felt all other important properties could be derived. These four basic properties are permeability, porosity, fluid saturation, and electrical conductivity. Some of the earliest work was done by Henry Darcy, a Frenchman. In 1856 Darcy, investigating the flow of water through a sand filter, equated flow rate with the cross sectional area of the conduit, the length, and the pressure head created by having the ends of the conduit at different levels. He found that a proportionality constant was required to balance the equality for various filters, and that the constant was a function of the sand packing. This constant became known as the permeability of the media. The unit of permeability is appropriately called the darcy, and is defined in the petroleum[5] industry as:

> A porous medium has a permeability of one darcy when a single phase fluid of one centipoise viscosity that completely fills the voids of the medium will flow through it under conditions of viscous flow at a rate of one cubic centimeter per second per square centimeter cross sectional area under a pressure or equivalent hydraulic gradient of one atmosphere per centimeter.

As can be seen from the definition, Darcy's law may be extended to include other fluids as well as water. Ideally, the permeability is a function only of the porous medium and not the particular fluid flowing. However, investigators such as Klinkenberg[6] reported permeability variations discovered when using gases as the flowing fluid. The permeability was found to vary with the gas pressure, and the variation was attributed to a phenomenon known as gas slippage. A discussion of this phenomenon is presented in most standard texts, e.g., Amyx, et al.[7]

Prior to discussing the work of Carman and Kozeny in the prediction of the permeability of a porous media from its known pore size distribution, we should discuss early work in the area of calculating porosity. The early investigations of Slichter[8] and Fraser and Graton[9] were concerned with the prediction of porosity of packings of uniform spheres. Since porous media are, in most cases, quite obviously not composed of uniform spheres, but of a myriad of particle sizes and shapes consolidated with every conceivable angularity, the work of these men must be enlarged upon to deal correctly with real materials. Whereas their work was largely theoretical, and others continued to work with predictions of porosity based upon grain size distribution utilizing seive analysis, the more recent work in this area has been through empirical determinations of porosity, by laboratory measurement, using such devices as the Kobe porosimeter[10] (mercury introduction),

or the Washburn-Bunting[11] porosimeter (air extracted by vacuum and measured).

Studies of pore size distribution have been carried out by several investigators including Carman,[12] using gas adsorption techniques; Ishkin and Kazaner,[13] passing two phase fluids through the media; Ritter and Erich,[14] using small angle x-ray scattering patterns, and by many others with variety of techniques. Grain size distribution work has been accomplished mostly by simple seive analysis and there is apparently a need for more work in this area. In fact, Masson[15] indicates that after extensively researching this field, there is "a definite need for an inclusive mathematical function containing a few definite parameters that would define a given size distribution for most conditions approaching ideal grinding."

One of the most interesting works done in the determination of pore size distribution is attributed to Nuss and Whiting,[16] who devised a method of impregnating core samples with plastic under high pressure. The plastic was then polymerized at a high temperature and the rock material removed by acid leaching. The remaining plastic reproduces the original pore space.

Many investigators have completed correlations between the porosity and permeability of porous media. Carman,[17] Kozeny,[18] and Wyllie[19] developed predictive equations from these correlations, whereby one property might be estimated

accurately from the other. In recent years in England, Collis-George and Childs[20] have refined the Kozeny Equation to improve results and allow greater generality. Francher, Lewis and Barnes[21] experimented with various porous matter to determine the relationship between grain size and fluid conductance. In their work they accounted for both laminar and turbulent flow. A brief discussion of the results is found in the text by Amyx.[22]

Nearly all rock samples brought to the surface contain amounts of liquid entrained in their pore space. The determination of the amounts and types of liquid is a very important characteristic description of the porous media. Comprehensive descriptions of the methods used to accomplish this determination are covered by Hill,[23] Horner,[24] Schilthius,[25] and others. The two most common techniques for determination of fluid content are by retorting, and by solvent extraction and distillation. Fluid saturation may also be determined by using an indirect approach, that is, by measuring some other property and correlating the findings with past data to estimate the fluid entrained within the sample. Use of electric logs and capillary pressure measurements are two such indirect means of arriving at the fluid saturation.

The fourth fundamental property of porous rock is the electrical conductivity. The rock, as it comes from the ground, is comprised of solid fragments, void space and some

amount of entrained liquid. Generally, the solids and the oil and gas components filling the pore spaces are non-conductors of electricity. The water, if it contains dissolved salts, then, is the only conductor. The ability or non-ability of a rock to conduct electricity is thus an indication of the interconnected or continuous water filled pores. Wyllie and Spangler[26] created the first model of porous media taking advantage of this phenomenon to characterize the media according to its electrical resistivity, and hence, derived such other properties as tortuosity and porosity. Cornell and Katz[27] and Wyllie and Gardner[28] added refinements in later models.

From the four basic properties covered, many other properties can be derived. Researchers in many fields have developed various descriptive properties which characterize media more explicitly for their particular use. For example, the geologist has expanded the characterization to include mineral content, while the petroleum engineer, because of his interest in flow properties, has developed such concepts as capillary pressure, relative permeability, and wettability. A description of these properties is beyond the scope of this report, but the bibliography references the works in this area of Purcell,[29] Leverett,[30] H. Brown,[31] Rose,[32] Slobod,[33] Fatt,[34] and Rapaport and Leas.[35]

Man has, then, taken advantage of many physical properties to descriptively classify porous media. However, in 1961,

A. E. Scheidegger and H. D. Fara,[36] writing in the Journal of Geophysical Research, point out that the usual geometrical properties, such as porosity and permeability, do not describe the media sufficiently for present requirements. Quoting their article:

> Geometrically, a porous medium is defined by giving the analytical equation of the surface which bounds its pore space. For any practical purpose, this is impossible to accomplish. Average geometrical quantities, such as porosity . . . , and specific surface . . . , are conceptually acceptable (and can be measured), but this is not true for the concept of "pore size." Since the pores form an extremely complicated system . . . , it is not easy to determine a pore size in a conceptually satisfactory fashion.
>
> Nevertheless, there is a great need for establishing a better geometrical characterization of a porous medium than is afforded by the mere determination of porosity and specific surface area.

Scheidegger and Fara desired a means of characterizing porous media through analysis of the statistical make-up of the media.

These investigators introduced four possible methods of statistical analysis: (1) by use of a correlation function, (2) a spectral analysis in terms of harmonic functions, (3) a spectral analysis in terms of other orthogonal functions, and (4) a spectral analysis of a specially constructed characteristic function of the media.

In each of the four techniques, the writers developed statistical data for interpretation by drawing an arbitrary line through the porous medium. The line has a designated origin and data points are selected at equally spaced points

along the line. At each of these points, a value of +1 is assigned if the line is, at that point, passing through void or pore space, and a value of -1 is assigned if the line is passing through solid. This series of +1 and -1 values, taken from the origin, define an orthogonal function f(s), which is characteristic of the line drawn through the function and of the media itself. If the medium is homogeneous and isotropic, and if the sample length is of sufficient size, theoretically, the functions or lines are independent of direction, origin, or position. The function is treated as a random function, as defined by Moyal,[37] and can be statistically analyzed in several ways.

If the function is defined as above for void and solid spaces, the mean of the function may be derived:

$$\overline{f} = +1(P) + (-1)(1-P) \tag{2.1}$$

where P = porosity or pore spaces

$$\therefore \quad \overline{f} = 2P-1 \tag{2.2}$$

In many cases, Scheidegger points out that it is inconvenient to work with a function having a mean other than zero, so he rewrites (2) as a new function with a mean that is equal to zero. The new function becomes:

$$\overline{f}'(s) = f(s) - (2P-1) \tag{2.3}$$

In the first of the four techniques which the authors suggest, the function is characterized by its autocorrelation function:

$$R(\Delta) = \lim_{S \to \infty} \frac{\int_{-S}^{+S} f(s)\, f(s+\Delta)ds}{\int_{-S}^{+S} ds} \tag{2.4}$$

Chapter III of this paper contains a discussion of the auto-correlation function.

The second technique referred to in the Scheidegger paper is to fit the function, f(s), with a Fourier series, and evaluate the Fourier coefficients, a_k and b_k. A suitable evaluation suggested is to use the equation:

$$c_k^2 = a_k^2 + b_k^2 \tag{2.5}$$

where c_k^2 is not only a fair representation of the medium's geometry (statistically), but has the added advantage of being independent of point of origin or phase angle.

The third and fourth techniques suggested are mere refinements of the two discussed briefly above. Each of the four techniques was offered by Scheidegger and Fara in an introductory manner, and there was very little in the way of tangible conclusions. However, these men opened an entirely new facet to the subject of characterizing porous media. Unfortunately, in correspondence between the author of the report in hand, and Dr. Scheidegger, the latter indicated that he had done no further work in this area. Further works in the refinement of these and similar techniques are now left to this author and others.

Perhaps brief mention should be included herein as to the history of statistical communication theory and time

series analysis, which Scheidegger has introduced into the world of porous matter. Granger[38] credits the astronomers as being the first to analyze time series, although he mentions discussions of meteorological and geophysical series in the middle of the nineteenth century. Even in these early discussions, it was noted that series of data taken over periods of time often follow patterns and contain certain trends, sometimes obvious and sometimes not so obvious. If the obvious trends such as those resulting from secular or seasonal variations could be subtracted from the series, the remaining series would contain random and independent data, and perhaps some of the not so obvious trends or patterned variations.

Many of the works in separating these "hidden periodicities" were instituted by La Grange,[39] Buys-Ballot,[40] Whittaker,[41] Stokes,[42] and Schuster.[43] In all of these works, the random independent series was assumed to be unimportant, and attempts were made to mathematically generate similar functions as those creating the hidden trends.

Schuster initiated the use of the periodogram, a method of pinpointing lesser periodic variations in a series. This approach was refined by such investigators as Whittaker and Robinson[44] who showed that apparent major trends may only be a combination of multiple lesser trends at their coincident peaks.

It followed that Kolmogoroff,[45] Weiner,[46] and Cramér[47] should refine these attempts at separating periodic tendencies

by using Fourier series analysis to segregate obvious sine or cosine trends. All of these earlier theoretical works led to practical applications, particularly in the field of communications by Tukey[48] and his assistants at the Bell Telephone Laboratories. Applications in other fields have followed, until today, time series analysis and communication theory are major tools of the business and economic world as well as the world of science and engineering.

CHAPTER III

HARMONIC ANALYSIS OF RANDOM FUNCTIONS

A simple means of analyzing a function is through a resolution into its harmonics by the use of a Fourier series. Where the function has one independent variable and one dependent variable, $f(x) = z$, the one dimensional or single Fourier series is appropriate. A two dimensional function with two independent variables and the one dependent variable, $f(x,y) = z$, requires the use of the double Fourier series. A discussion of these mathematical tools is contained in Appendixes A and B. Where random processes or random functions are being considered, the classical harmonic analysis is not directly applicable. However, it has been shown that a "type" of Fourier analysis is still possible. This theory, which has grown out of statistical theory of communication, provides the engineer with additional tools to assist in analyzing random phenomena.

Two very important properties of a signal or function, which for our purposes serve to characterize the function, are the power spectrum and the autocorrelation function. Prior to presenting the mathematical background for calculating these properties, it may be well to offer a laymen's definition of each.

A. Power Spectrum

This is a term used in basic communication theory which has been borrowed from the field of electricity. Various analogies have been drawn to describe the difficult concept of spectral analysis, one of the better ones being that discussed in Granger and Hatanaka.[49] The author will attempt to draw an analogy of similar nature. If a person stands on the corner of 42nd Street and Broadway, during a very busy period, his ears will pick up many and varied sounds. Each of these sounds will appear to be different, although in many cases the "sound" which one hears is in reality a combination of many sounds coming from a range of signal creating devices.

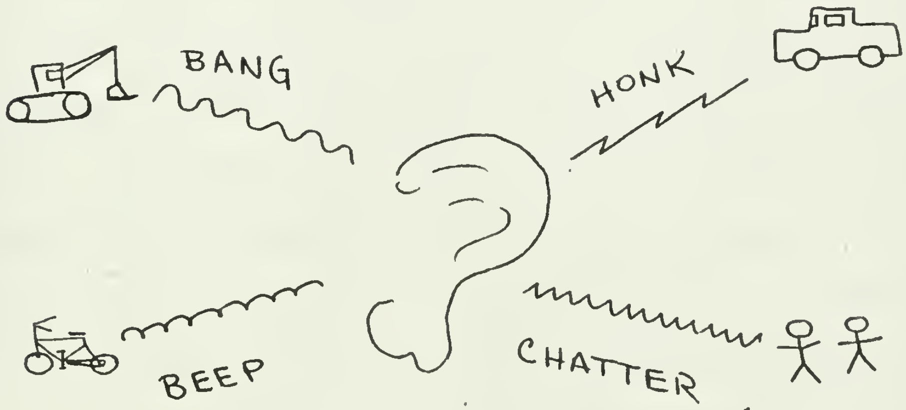


Figure. 3.1 The Human Ear Performing Simple Spectal Analysis

The different sounds are a result of many signal generators, emitting different frequencies. The human brain is equipped with a device, not unlike a filter used in electrical circuits, which isolates each signal and attempts to analyze it. Sometimes it is successful and reports, "That is an automobile horn, and I must get out of the street," and sometimes the cacophony is so great that the individual signals can not be separated and interpreted. The brain is performing a type of spectral analysis when it selects, from a number of frequencies that the ear is hearing, individual signals and classifies them according to amplitude and frequency. Of course, in communication theory, and perhaps even in our analogy, there may be so many frequencies that it is only within our capability to identify and classify within ranges of frequencies. This is more often the case where a receiver is obtaining both intended or predetermined signals and "noise" which is added by some unknown and random generator. The mathematical process of decomposing a composite signal into its component frequencies and determining the proportion of the total signal at each frequency has been called Power Spectrum Analysis, or it is often abbreviated as Spectral Analysis.

Spectral analysis has been used more and more in recent years to analyze economic time series for the business world. Brown[50] describes the use of this theory to separate the many factors which influence an economic series, such as

seasonal and secular trends. Here, as in electrical and communication work, we discover the presence of "noise," or factors which we cannot interpret as to origin, or predict as to future behavior.

B. Autocorrelation Function

The autocorrelation function may be described as the new function created when a function is correlated with itself. However, in order to provide a more meaningful and definitive picture, we must look at the definition shown in Ekeblad.[51] This text points out that the degree of autocorrelation in a time series indicates how well we may predict some as yet unknown future point in the series from the data available. A very simple analogy can be given. If the sales in a certain business fluctuate from month to month as shown in Figure 3.2. and we compare each month's sales with the preceding month's sales, we can see that following every high sales month there is a lower sales month, and vice versa.

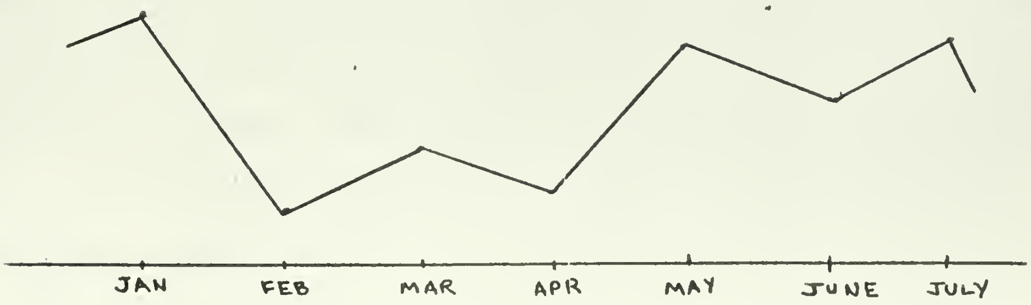


Figure 3.2 Sales Function

It is very simple to predict that in the future for every high month, there will be a lower month to follow. However, if we increase our "lag" and compare only every second month, we find that it is harder to make such predictions. If we go further and compare only the third months, we lose even more confidence in our predictions. This is a very simplified explanation of Figure 3.3 which generally depicts the autocorrelation function.

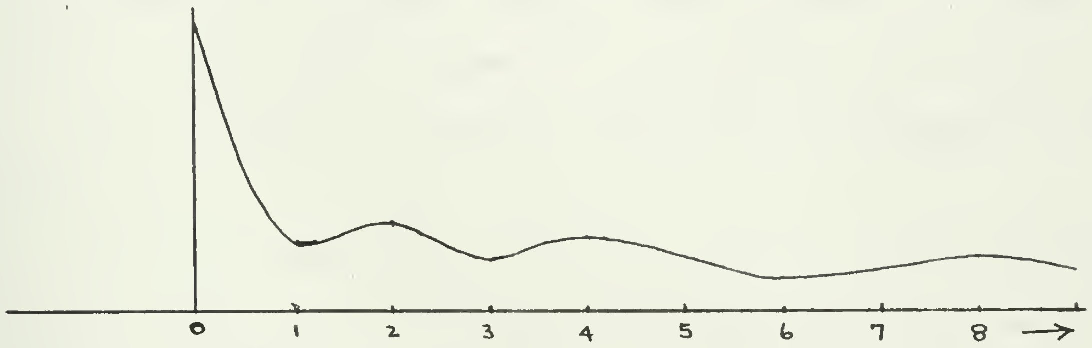


Figure 3.3 Autocorrelation Function

Figure 3.3 shows that the autocorrelation function is maximum at a minimum lag and generally decreases as the lag is increased. Relative maxima in the function appear which correspond to basic cycles existing in the data, but unless the function is periodic, i.e., repeats itself identically at some latter period, the value of the autocorrelation function will never approach the value determined at zero lag. Likewise, as the lag increases, our confidence of prediction decreases.

While we will not make use of spectral analysis or the autocorrelation theory to separate and analyze signals, make predictions, or even use the correlation properties of the latter, we will attempt to take advantage of the properties of each to characterize or represent functions which are generated by our porous-media samples. For example, the autocorrelation function has a property of retaining all of the harmonics of a given function, containing no new ones, and discarding the phase angle. The usefulness of this property shall become apparent as we explain the application of these statistical methods to the work of characterizing porous media.

With the background definitions of the power spectrum and the autocorrelation theory, we may now show the mathematical derivation of each.

C. Periodic Functions

Lee[52] shows that a periodic function may be expressed as a Fourier series, as discussed in Appendix A.

$$f(x) = \frac{a_o}{2} + \sum_{n=1}^{n=\infty} a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L} \tag{3.1}$$

where:

$$a_n = \frac{2}{L}\int_{-L/2}^{+L/2} f(x)\cos\frac{n\pi x}{L}\,dx \tag{3.2}$$

$$b_n = \frac{2}{L} \int_{-L/2}^{+L/2} f(x) \sin \frac{n\pi x}{L} \, dx \qquad (3.3)$$

L = Fundamental period of the function.

x = value of the independent variable at equally spaced points along the function, such that x goes from -L/2 to L/2.

n = number of harmonic terms to which the series is expanded.

Equation (3.1) may be rewritten:

$$f(x) = \sum_{n=-\infty}^{n=\infty} F(n) e^{jn\pi \frac{x}{L}} \qquad (3.4)$$

in which:

$$F(n) = \frac{1}{2} (a_n - jb_n); \text{ for } n = 0, 1, 2, \ldots \qquad (3.5)$$

j = the imaginary operator, equal to the square root of minus (negative) one. (Some texts use the notation "i," and a more definite description may be found in any text dealing with complex variables.)

combining (3.2) and (3.3)

$$F(n) = \frac{1}{L} \int_{-L/2}^{+L/2} f(x) e^{-jn\pi \frac{x}{L}} \, dx \qquad (3.6)$$

Equation (3.6) is the Fourier Transform of the periodic function. Lee shows that this equation represents an "analysis" of the amplitudes and phase angles of each sinusoid into which function, f(x), is resolved. F(n) is also called the complex spectrum for this reason.

To separate the amplitude and phase characteristics of the complex spectrum, F(n), we may write it as:

$$F(n) = \frac{1}{2}\sqrt{a_n^{\,2}+b_n^{\,2}}\ \exp\left[j\tan^{-1}\left(-\frac{b_n}{a_n}\right)\right] \tag{3.7}$$

where the amplitude is depicted by $|F(n)|$:

$$|F(n)| = \frac{1}{2}\sqrt{a_n^{\,2}+b_n^{\,2}} \tag{3.8}$$

and the term θ_n, indicates the phase angle:

$$\theta_n = \tan^{-1}\left(-\frac{b_n}{a_n}\right) \tag{3.9}$$

Combining the above, Equation (3.4) may be written:

$$f(x) = \sum_{n=-\infty}^{n=\infty} \frac{1}{2}\sqrt{a_n^{\,2}+b_n^{\,2}}\left[\cos\left(\frac{n\pi x}{L}+\theta_n\right)\right] \tag{3.10}$$

To analyze possible correlation between two functions, we use the function, $\varphi_{1,2}$, where:

$$\varphi_{1,2}(\tau) = \frac{1}{L}\int_{-L/2}^{+L/2} f_1(x)\, f_2(x+\tau)\, dx \tag{3.11}$$

We call the left-hand member of Eouation (3.11) the correlation function at a given lag (τ). By multiplying one function by the other, at various values of τ, we can determine the points of best correlation.

If we adopt the special condition, where $f_1(x) = f_2(x)$, we can write (3.11) as:

$$\varphi_{1,1}(\tau) = \frac{1}{L} \int_{-L/2}^{+L/2} f_1(x)\, f_1(x+\tau)\, dx \qquad (3.12)$$

Lee shows that the autocorrelation function is also equat to:

$$\sum_{n=-\infty}^{\infty} \left|F(n)\right|^2 e^{jn\pi \frac{x}{L} \tau} \qquad (3.13)$$

When τ equals zero, Equation (3.13) reduces to:

$$\varphi_{1,1}(\tau) = \frac{1}{L} \int_{-L/2}^{+L/2} f_1^{\,2}(x)dx = \sum_{n=-\infty}^{\infty} \left|F_1(n)\right|^2 \qquad (3.14)$$

or the mean square value of the function equals the sum, for all harmonics, of the square of the absolute value of the spectrum. The power spectrum is then defined by:

$$\Phi(n) = \left|F_1(n)\right|^2 \qquad (3.15)$$

It can be demonstrated that the power spectrum and the autocorrelation function are Fourier transforms of each other. As previously stated, all functions consisting of identical sinusoidal components have the same autocorrelation function, regardless of phase angle. Thus, the power spectrum, as a Fourier transform of the autocorrelation function, is likewise independent of initial phase angle and having determined the latter, the former is uniquely determined.

Lee also shows that these functions may be written in terms of Fourier coefficients:

If $\left|F_1(n)\right| = \frac{1}{2}(a_n + jb_n)$

then:

$$\phi(n) = \left|F(n)\right|^2 = \frac{a_n^{\ 2} + b_n^{\ 2}}{4} \tag{3.16}$$

and:

$$\mathcal{C}_{1,1}(\tau) = \frac{a_o^{\ 2}}{4} + \frac{1}{2}\sum_{n=1}^{n=\infty}\left(a_n^{\ 2} + b_n^{\ 2}\right)\cos\frac{n\pi}{L}\tau \tag{3.17}$$

From Equations (3.16) and (3.17), it is possible to calculate these desired properties of a function with the use of a computer program. (See Appendix C.)

D. Aperiodic Functions

Whereas the tool of analysis for handling a periodic function is the Fourier expansion, it is necessary to rely upon the Fourier integral to analyze the aperiodic function. The derivation of this integral and the steps necessary to evaluate the power spectrum and autocorrelation function for this type of function are not required within the scope of this paper. A discussion of the theory is contained in texts, such as Tolstov[53] or Lee.[54]

E. Random Functions

We define a random function as follows:

1. The generating process cannot be completely evaluated and a precise and unique determination of the value of the dependent variable for any value of the independent variable cannot be made.

2. As a consequence of (1), the value of the dependent variable cannot be predicted at any given future point in time.

In other words, a random function is one that is generated by some unknown source which dictates the manner in which the function behaves, but, due to its stochastic or probabilistic nature, cannot provide the actual value of the function at any given moment in time. An example of a random function might be the noise that a radio picks up in addition to the intended message or program. This noise may be the results of thermal emissions, static, or spillover from other frequencies, but an exact description of the noise is not possible because of the complexity and random nature of the noise source. It might be noted that in most economic series, although containing obvious trends caused by seasonal or secular variations which may be isolated, certain strong random components exist.

Because of the problems created by this "noise," it is of great import in communications work that analysis of such functions be a possibility. In order to analyze random functions, we find it necessary to introduce certain restrictions, and assumptions. First, we assume that the function has an infinite period and that during this period, the function will trace a pattern, such that the value of the dependent variable will pass through, or infinitely close to, all possible values which are consistent with

the "ergotic hypothesis" as espoused by Maxwell, and discussed in Lee. Accepting this theorem, we may consider our random generating process as generating a group of finite length random functions rather than one infinitely long function having the combined characteristics of the entire grouping. The individual functions follow the nature of the infinite function on an average value basis and in a probabilistic manner. By the latter, it is meant that such properties as mean value crossings, or zero value crossings per unit length follow the probability distribution of the parent function. These finite functions will be referred to as an ensemble of random functions. They are all generated by the same source. This grouping of the ensemble is a very necessary step in the analysis of random functions, as realized and set forth by Gibbs.[55]

In such an ensemble of functions, each of the finite functions will be called member functions. It is important to realize that we assume that each of these member functions is generated by the same source, although we cannot mathematically formulate the operation of the source. This being so, we sense that properties which characterize any member of the ensemble, in turn characterize all others, as well as the infinite function from which the ensemble was created. One of these properties would be the autocorrelation function,

$$\varphi_{1,1}(\tau) = \lim_{L\to\infty} \frac{L}{2} \int_{-L}^{+L} f_1(x) f_1(x+\tau)\,dx \qquad (3.18)$$

which we assume exists for all values of τ. Another such property is the power spectrum. Wiener[56] developed the theorem that the autocorrelation function of a random function and the power density spectrum are related to each other as Fourier Cosine transforms. The fundamental assumption underlying this theorem is that the autocorrelation function exists and is common for all member functions of the ensemble. Blackman and Tukey[57] state that, although Equation (3.18) is frequently called the autocorrelation function, a more appropriate name for the term defined by Equation (3.18) is the autocovariance function. They state that the autocorrelation function should be defined as the normalized autocovariance function obtained as the ratio of the latter function at any lag to the function at zero lag. This affords the autocorrelation function the property of having the value of unity at zero lag. However, it is convenient for our purposes, and entirely consistent with the definitions, to use the names interchangeably for characterizing porous media.

In order that convenient equations might be developed which will allow us to calculate these properties with the aid of a computer, we refer to Granger and Hatanaka.[58] They show that for real processes the autocovariance, μ_τ, may be written:

$$\mu_\tau = 2\int_0^\pi \cos \tau\omega \, f(\omega)d\omega \qquad (3.19)$$

where: $\omega = \frac{\pi x}{L} \leq 0$

Formal inversion provides the power spectrum

$$f(\omega) = \frac{1}{2\pi}\left(\psi_o + \sum_{j=z}^{\infty} \psi_j \cos j\omega\right) \tag{3.20}$$

For finite data, using numerical integration, an approximation to Equation (3.19) is:

$$c_\tau = \frac{1}{n-\tau} \sum_{i=1}^{n-\tau} \left(x_i - \bar{x}\right)\left(x_{i+\tau} - \bar{x}\right) \tag{3.21}$$

where n = number of data points

The estimate of Equation (3.20) becomes

$$\hat{f}(\omega) = \frac{1}{2\omega}\left(c_o + 2 \sum_{j=1}^{n-1} c_j \cos j\omega\right) \tag{3.22}$$

Hannan[60] shows that this is a poor estimate, but that it may be improved by properly weighting the values of the autocovariance terms, so that

$$\hat{f}(\omega) = \frac{1}{2\pi}\left(c_o\lambda_o(\omega) + 2 \sum_{j=1}^{m-1} c_j\lambda_j(\omega) \cos j\omega\right) \tag{3.23}$$

where "m" is arbitrarily selected by the user and denotes the number of lags used. It should be less than one-third the value of "n." $\lambda_i(\omega)$ represents the value of the weighting factors.

There is considerable discussion concerning the choice of the weighting factors, $\lambda(\omega)$. However, the two most widely used, developed by J. W. Hamming and J. Von Hann,

are very similar and for this paper we will use the latter's system, now commonly known as the Tukey-Hanning estimate. It has the values

$$\lambda = \frac{1}{2}\left[1 + \cos\frac{\pi\tau}{m}\right] \qquad (3.24)$$

The actual equations for estimating the covariance and the power spectrum may now be written

$$c_\tau = \frac{1}{n-\tau}\left[\sum_{x=1}^{n-\tau} x_i x_{i+\tau} - \frac{1}{n-\tau}\sum_{i=1+\tau}^{n} x_i \sum_{i=1}^{n-\tau} x_i\right] \qquad (3.25)$$

$$L_j = \frac{1}{2\pi}\left[c_o + 2\sum_{\tau=1}^{m=1} c_\tau \cos\frac{\pi\tau j}{m} + c_m \cos \pi j\right] \qquad (3.26)$$

$$U_j = .25\, L_{j-1} + .5\, L_j + .25\, L_{j+1} \qquad (3.27)$$

where $L_{-1} + L_{+1}$, $L_{m+1} + L_{m-1}$.

The L_j terms are the raw estimate of the spectrum, and the U_j terms are the smoothed values. The smoothing operation of Equation (3.27) is apparent from the form of the equation.

To illustrate Equations (3.25) and (3.26), we might examine two very simple functions. Function A is an on-off function as shown in Figure 3.4.

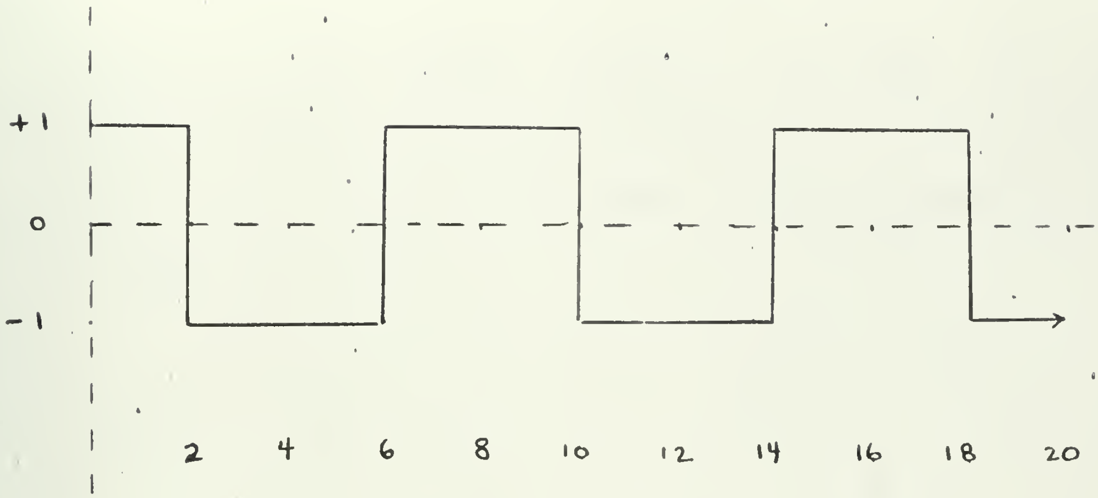


Figure 3.4 Example Function A

By examining the figure, it is seen that the function repeats every eight units and that the autocovariance function has the shape shown in Figure 3.5.

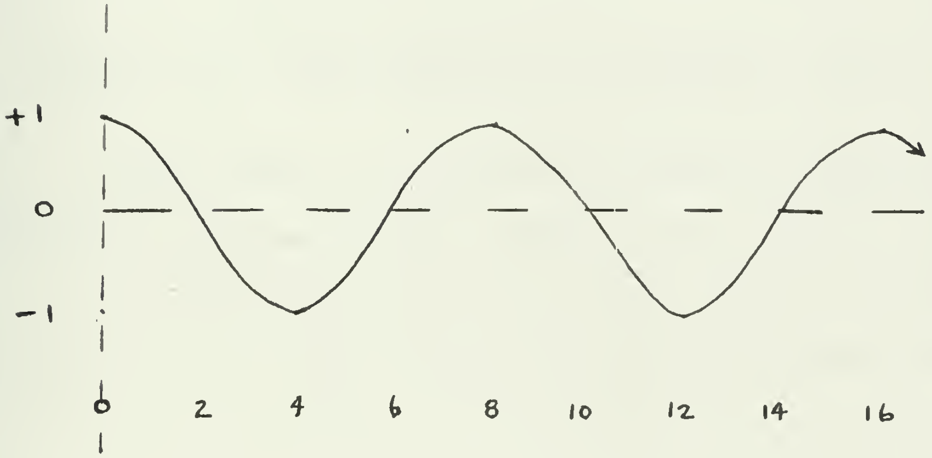


Figure 3.5 Autocovariance Estimate of Function A

Referring to Equation (3.26), it can be seen that the power spectrum will have a peak value when the frequency parameter "j" is such that the cosine term, cos $\frac{\pi\tau j}{m}$, has a period coincident with the period of the autocovariance

function. As an example, taking m = 150 lags, the power spectrum reaches its peak value when the frequency parameter "j" is equal to 37.5. The power spectrum appears as shown in Figure 3.6.

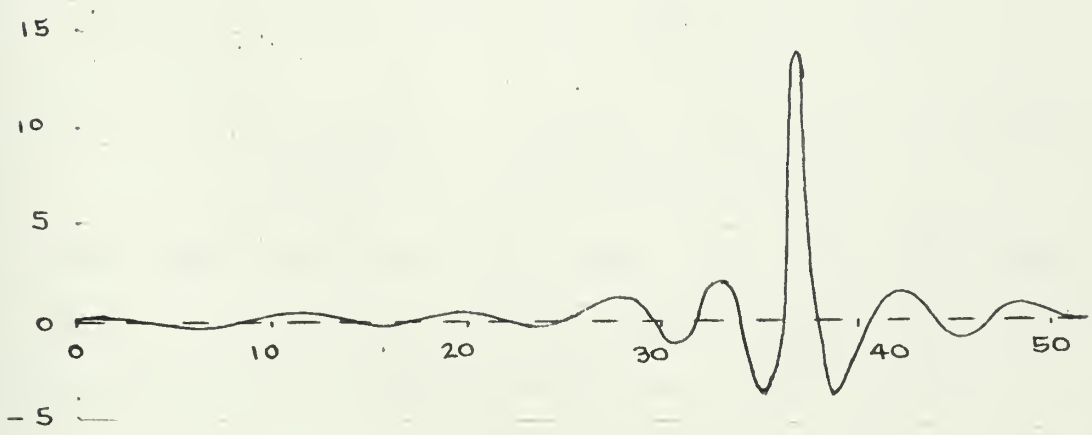


Figure 3.6 Power Spectrum of Function A at 150 Lags

These results are easily checked by observing that the original function passes through 37π cycles in 150 units.

A second function which can be used to demonstrate the nature of the analysis performed by use of the auto-correlation function and the power spectrum is Function B. Function B is a combination of two simple on-off functions which have been superimposed. The frequency of the first function is four cycles per eighty points. The frequency of the second function is ten cycles per eighty points. Function B, as shown in Figure 3.7, has very little apparent regularity.

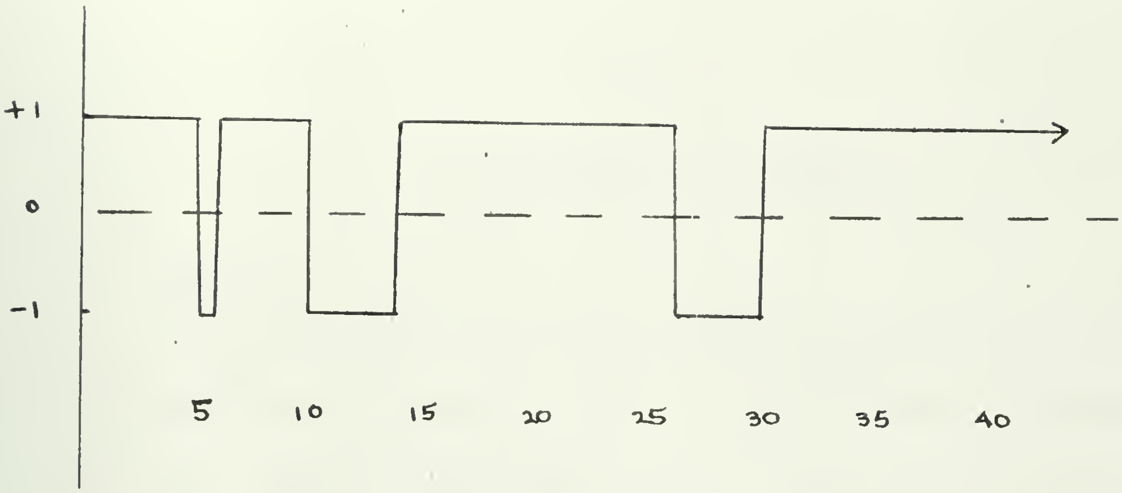


Figure 3.7 Example Function B

The power spectrum for 50 lags shows peaks at 5 and 12.5, which were the individual peaks for each basic frequency used in the combined function. In other words, Equation (3.26) effectively selects the fundamental frequencies which constitute a function, even though they may not be apparent individually. Further, a measure of the relative contribution of each frequency is offered.

The computer program "COVAR," discussed in Appendix C, has been written specifically to perform the mathematical operations set forth in Equations (3.25) and (3.26). The above figures were made from data and results of that program, run on Functions A and B.

CHAPTER IV

EXPERIMENTAL PROCEDURE

A. General

In this thesis an attempt is being made to classify or characterize porous media. It is our goal to differentiate between media which are not statistically identical through the use of various forms of statistical analysis. In order to apply the various tools outlined in Chapter III, we require a variety of samples which we know to be disimilar. In this chapter, we will discuss the selection and preparation of these samples and the collection of data therefrom.

B. Preparation of Samples

Data used in this thesis were taken from two types of porous-media samples. The first sample was a photomicrograph of an actual porous medium. The other samples used were prepared by the author by imbedding multi-size spherical glass beads in plastic.

The photomicrograph is a photograph of an actual porous medium taken through a microscope. In the sample used, the photographic image was enlarged ninety-two times from an original size of approximately .12 centimeters by .16 centimeters. This photograph was taken from the paper by Fara and Scheidegger.[60] The original medium was sandstone with a porosity of 26.3 percent.

The other samples used were created by the author in the following manner:

Materials:

(1) Laboratory glass beads, three sizes: 3mm., 4mm., 5mm.

(2) Natco Premium Quality Clear Polyester Casting Resin, NL-600.

(3) Violet dye (water base).

(4) Waxed paper cups.

Procedure:

(1) Mix dye and resin to achieve a colored, slightly opaque plastic.

(2) Place liquid plastic in waxed paper container.

(3) Drop glass beads, in varying proportions as required, into the plastic.

(4) Place in over for one day (30 degrees centigrade). Allow to set.

(5) Cut through sample in a random manner to realize an unbiased cross sectional surface of the mold. This surface area represents a model of a porous medium. Where the glass beads are apparent is taken as solid space, and where plastic is apparent is taken to be pore space.

For this paper, three samples were prepared. Sample X consisted of 3mm. beads only. Sample Y was a mixture of equal volumes of 4mm. and 5mm. beads. Sample Z was a mixture of equal volumes of 3mm., 4mm., and 5mm. beads. The cut samples were approximately two inches in diameter.

C. Preparation of Data

Following the preparation of the samples, photographs of each were made by the Photo and Graphic Arts Department, University of Kansas (see Appendix D). The images of the created samples were expanded six times, while that of the Scheidegger photomicrograph was already of appropriate dimensions. An 80 x 80 grid, with ten units per inch was overlaid on each photograph. At each grid point, the data in point was recorded as being a +1 if a pore space (plastic), and a -1 if a solid space (beads). These 80 x 80 data points were easily converted to input information on eighty column computer cards, for use with the programs written for this thesis. (The negative values were read into the computer as fixed point zeros and changed internally to their correct values as negative ones.)

CHAPTER V

DISCUSSION OF RESULTS

The mission of this project was to provide means of statistically analyzing porous-media, to show statistical variation in media known to be different, and to offer direction and inspiration to further work in this area. The author fully realizes that from the limited number of samples dealt with in this thesis, there is insufficient evidence to prove that the far reaching goal of statistically characterizing all media can be accomplished. However, in this chapter, we will offer indications that the above outlined missions have been successfully completed and that success of the overall goal mentioned is entirely possible. It is hoped that from the results shown, others will be stimulated to continue research toward that end.

Two basic computational tools were used to statistically analyze the data taken from the porous-media samples. Although discussed in Chapter IV, it might be well to point out again that these data were taken at equally spaced points along lines randomly drawn across the cut-face of the samples. Where the line passed through pore space, a value of +1 was assigned to the function, and where the line bisected solid, a -1 value was assigned. A series of these points taken in

order along the line then formed a simple orthogonal function with the +1 to -1 cycle arranged in a seemingly random pattern. Based upon the assumption that porous media are statistically constituted, it was hoped that several of these lines or functions taken from the same medium would be statistically similar while those taken from different media would vary by what is known as a "significant difference."

The first method used to statistically analyze the functions was to fit each with a single Fourier series using "1D Fourier" computer program. (See Appendixes A and C.) Although each function was considered as being random in nature, it was analyzed over a finite length. The Fourier coefficients were determined and then combined according to the Lee estimate of the power spectrum, $c_n^2 = a_n^2 + b_n^2$, so that the initial starting point or phase angle might be disregarded. These values of c_n, as estimates of the power spectrum, are measurements of the contributions of the fundamentals frequencies constituting each function. This is essentially the technique used in the paper by Fara and Scheidegger.[61]

Typical results of the single Fourier series approach are displayed in Table 5.1 Shown are the c_n values for four lines across the same media. As can be readily observed, there is very little consistency among the values, indicating that this method yields a poor estimate of the power spectrum. As a result, the simple Fourier approach was not

pursued to any great detail, although later in this chapter we will compare the results of using this approach with those of our second technique on an identical sample.

Because of the poor results achieved in the single Fourier approach, no analysis of the two dimension cross sectional faces of the media was attempted. However, a computer program "2D FOURIER" to determine double Fourier series coefficients for a surface was developed, tested, and is fully operational. (See Appendixes B and C.)

The second statistical tool used to analyze the data was the determination of the communication theory properties of the functions or lines. These properties, the autocovariance function and the power spectrum, are discussed in Chapter III. The properties were calculated using Fortran equivalents of Equations (3.25) and (3.26) in the computer program "COVAR." (See Appendix C.) As a part of that program, the power spectrum was plotted for each of several lines per media type, as well as the 90 and 95 percent confidence bands. These bands were created by multiplying the smoothed power spectrum by constants presented in Granger and Hatanaka.[62] The constant used is a function of the "equivalent degrees of freedom," an approximation chosen to allow for the fact that the series under consideration is not necessarily exactly normally distributed. These bands indicate 90 or 95 percent confidence that the true value of the power spectrum lies within that range of values. In

other words, 90 or 95 percent, as the case may be, of all power spectra values from the same population should lie within that range. Generally, the approach was to compare data lines from the same media to see if the observed plots of the power spectra were similar. Then it was determined whether the values of each function within the sample, at each frequency, fell within the confidence bands of the other lines within that sample or type of media. A similar technique was used to compare the power spectra profile of one sample media with that of another media.

For these comparisons, the effect of sample lengths was investigated. Samples of various lengths were created by combining the eighty point crossings developed in accordance with Chapter IV. The crossings were shuffled to reduce bias and eliminate peaks which would occur in the autocovariance function at eighty point intervals. These peaks would be due to the similarity between adjacent lines drawn across a porous media. It was discovered after a series of trials that the longer the lines, the less bias there was to the idiosyncrasies of single crossings, i.e., a line might pass through continuous solid or pore space for an many as twenty or thirty points without interruption or change. Longer lines tend to minimize the effect of these rare exceptions. As a result, lines of 3200 data points or 40 crossings were utilized, this being the maximum number of points that could be handled on the computer at one time. It might be noted that lines of 1600 data points afforded

very similar results, indicating that the lines of 3200 points were of sufficient length.

It was discovered that the number of lags selected, or more properly stated, the length of the maximum lag selected, played a very important part in the analysis. Again after many trials, it became apparent that when the maximum lag selected was too short, the apparent frequencies were "packed" at the very low power spectrum values. In other words, with many frequencies constituting the function, each band or grouping of frequencies was too wide or contained too many frequencies to effectively delineate the function. To long a maximum lag had the effect of causing random spikes in the power spectrum which were characteristic of an individual line but not of the media as a whole. These "spikes" might be considered a result of sampling variability. Also, at long lags, values of the autocovariance function tended to distort the power spectrum profile. In light of the above, a lag of 40 was selected as being optimum. It had the effect of smoothing the data and still offering good disorimination.

Figures 5.1 through 5.3 offer three plots of the power spectra of different media. Appendix E contains the computer results from which these lines were plotted and the computer plots. The three profiles are obviously different, but it is important to discuss these differences in a more absolute statistical manner. As shown in the plots, the

95 percent confidence band of one of the lines is superimposed upon the figure. By looking at each figure, it can be seen that generally the power spectrum of the other, or "outer" line lies within that confidence band. More specifically, the outer line is within the band at 24 points out of 40 in Figure 5.1. This is the equivalent of 60 per cent of the time. In Figure 5.2 the result is 23 out of 40 (57.5%) and in 5.3 it is again 23 out of 40 (57.5%).

While these results violate the 95 percent criteria, it is important to note that many of the points are very close to the limits and as only two examples were displayed per sample, variability of samples has a great effect. Ideally, many more lines per sample should be obtained, plotted and analyzed. Then the confidence bands around the mean of those lines should be superimposed upon a plot of the lines. Looking at all of the power spectra values that lie within these bands, the 95 per cent criteria would be approached. Using this method on four lines of shorter length from these same media, results very close to that criteria were realized and as more samples are considered, it is expected that the 95 percent criteria would be fulfilled. However, this was not proven in this work. By the time the tools to analyze media had been developed, and made available in the form of fully tested computer applications, and when the results discussed above became apparent, there was insufficient time to create or acquire additional samples for this thesis.

Cross comparing samples, there is very little correlation in comparison to the correlation within the same media. Between Figures 5.1 and 5.3, which are very similar media (see Chapter IV), the base lines are within the other's confidence bands only 12 of 40 points, or 30 percent of the times. Between Figures 5.1 and 5.2, the result is only one out of 40 (2.5%), and between 5.2 and 5.3, it is one out of 40 (2.5%). Visually comparing the different spectra, it can be seen that the 3mm. beads sample, which only has small beads in its make-up, has lower values than the other samples at low frequencies, and higher values than the others at higher frequencies. This is to be expected as there is a greater fluctuation in the +1 to -1 cycle in the original 3 mm. bead data. There is less opportunity for long strings of +1 or -1 values due to this composition of the sample. Similar analysis can be made between the other samples.

It is felt then, that this is a definite indication that the general shape of the power spectrum shown is indicative of the composition of the media from which the data is taken. Spectra from different media are not only visibly different, but also it is noted that the differences appear to be statistically significant. Again it is repeated that many more samples of various types need to be analyzed to prove this conclusively, but the technique is readily available, and indications are that the results would be favorable.

Of interest, Table 5.2 and Figure 5.4 display the computed autocovariance estimate and the power spectrum of the actual rock sample discussed in Chapter IV.

A brief comparison was made between the two techniques used to analyze the functions, the single Fourier series and the power spectrum. Figure 5.5 shows this comparison. The function analyzed was the line drawn across the porous media shown in the Fara-Scheidegger paper. (See Chapter IV). As can readily be observed, there is very little consistency in the results. This is further indication that the single Fourier series approach is not particularly suited for this type of analysis.

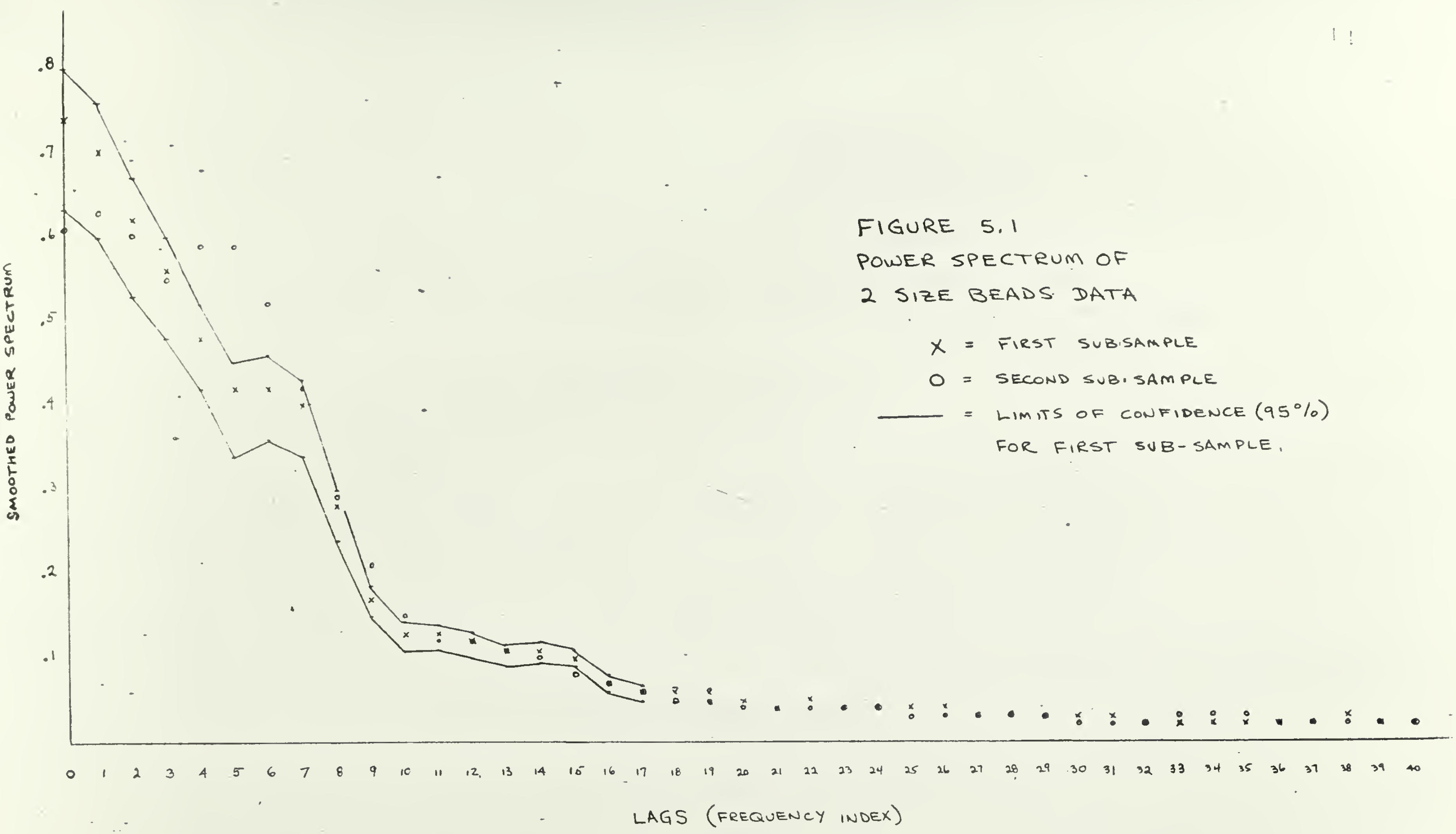


FIGURE 5.1
POWER SPECTRUM OF
2 SIZE BEADS DATA

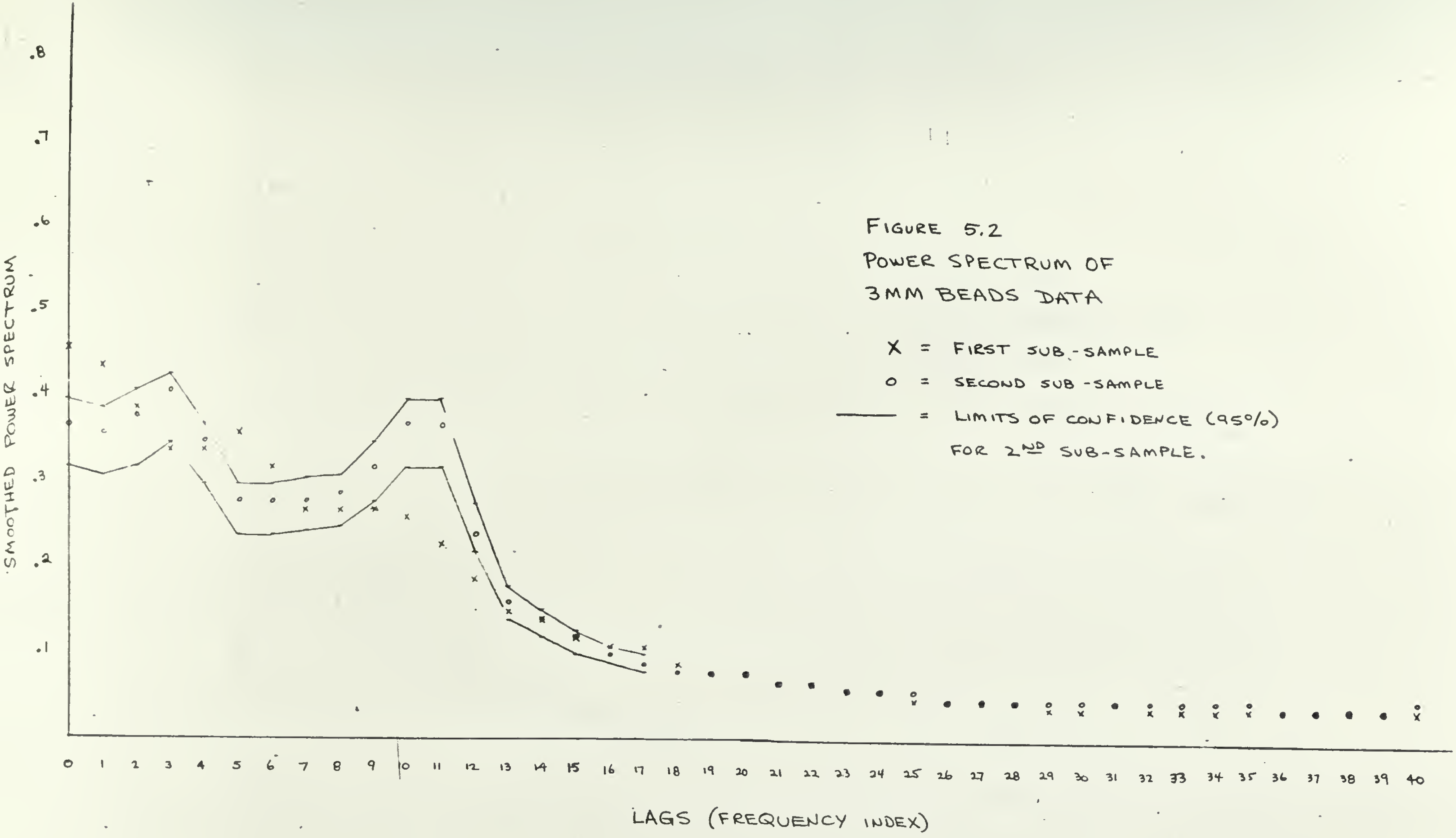


FIGURE 5.2
POWER SPECTRUM OF
3MM BEADS DATA

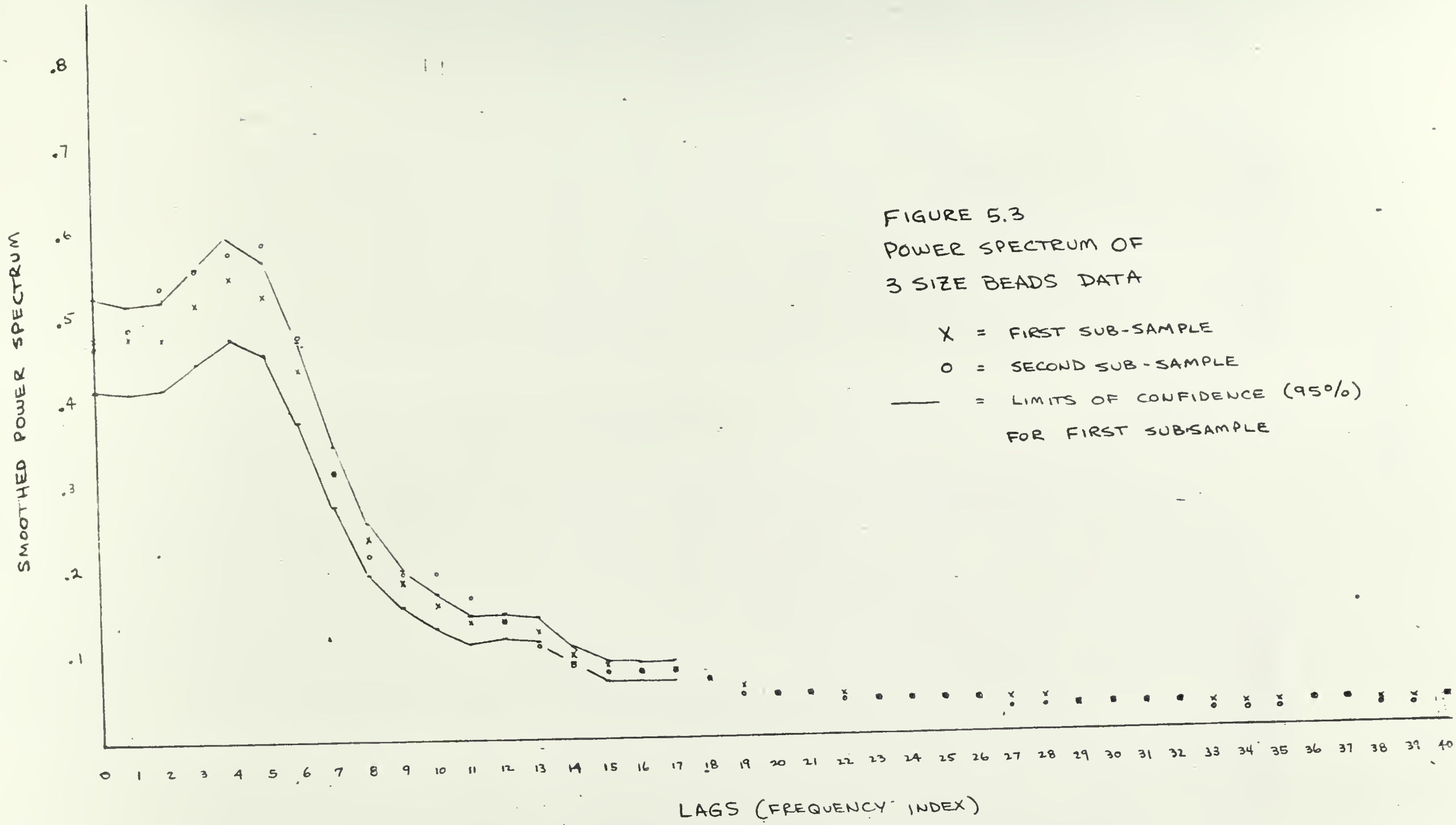
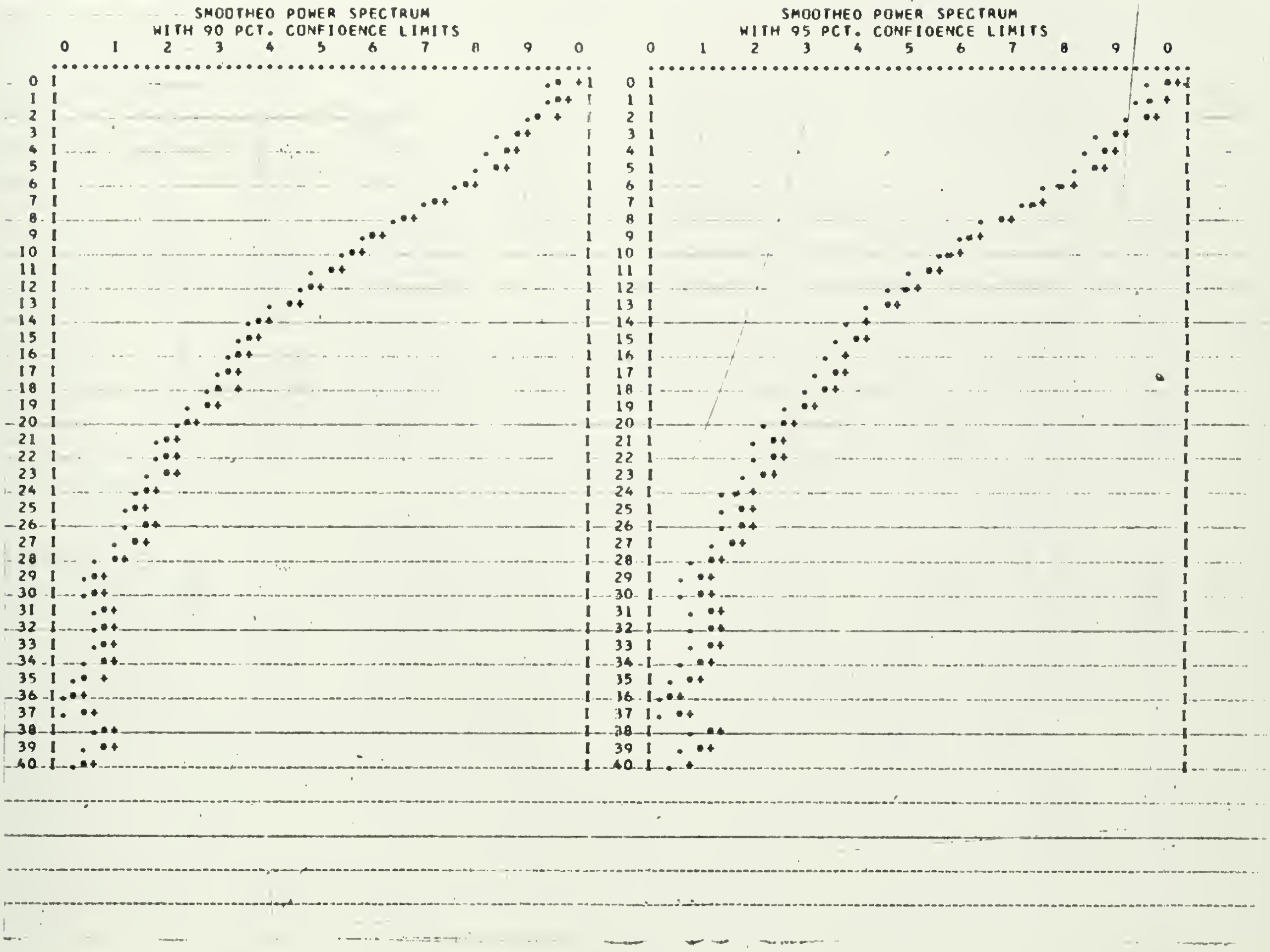


FIGURE 5.3
POWER SPECTRUM OF
3 SIZE BEADS DATA

FIGURE 5.4

POWER SPECTRUM ESTIMATION
FOR DISCRETE TIME SERIES
USING TUKEY HANNING WEIGHTING FACTORS

SCHEIDEGGER DATA 3200 POINTS LAG = 40



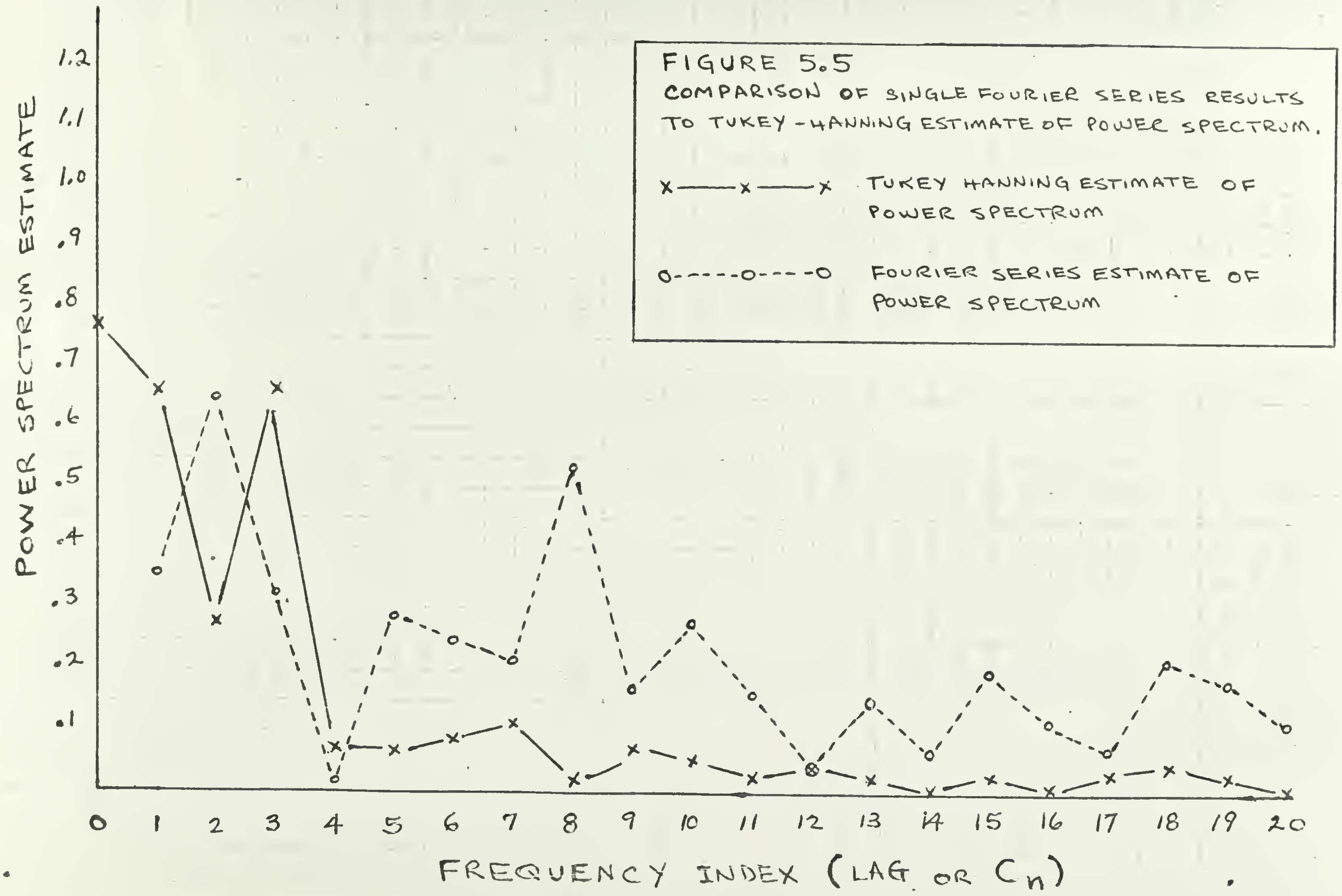


FIGURE 5.5
COMPARISON OF SINGLE FOURIER SERIES RESULTS TO TUKEY-HANNING ESTIMATE OF POWER SPECTRUM.

TABLE 5.1

Tabulated Results of Single Fourier Series Analysis of Three Size Beads Data

Power Spectrum Estimate	1	2	3	4
C(0)*	.2925	.5637	.4425	.2900
C(1)	.0005	.0012	.0048	.0298
C(2)	.0033	.0002	.0039	.0029
C(3)	.0057	.0000	.0001	.0029
C(4)	.0020	.0002	.0007	.0006
C(5)	.0078	.0002	.0017	.0023
C(6)	.0002	.0002	.0017	.0023
C(7)	.0028	.0002	.0023	.0140
C(8)	.0058	.0000	.0041	.0046
C(9)	.0022	.0109	.0018	.0036
C(10)	.0027	.0028	.0017	.0005
C(11)	.0020	.0011	.0151	.0071
C(12)	.0012	.0016	.0008	.0018
C(13)	.0030	.0000	.0080	.0050
C(14)	.0005	.0006	.0040	.0001
C(15)	.0034	.0003	.0049	.0018
C(16)	.0049	.0000	.0012	.0024
C(17)	.0048	.0007	.0017	.0079
C(18)	.0049	.0003	.0007	.0017
C(19)	.0202	.0006	.0087	.0187
C(20)	.0003	.0120	.0048	.0056
C(30)	.0024	.0124	.0012	.0015
C(40)	.0047	.0139	.0076	.0039
C(50)	.0119	.0066	.0148	.0010

*C(0) is an indication of the porosity of the media. The actual porosity of the sample was 35.65 percent.

TABLE 5.2

POWER SPECTRUM ESTIMATION
FOR DISCRETE TIME SERIES
USING TUKEY HANNING WEIGHTING FACTORS

SCHEIDEGGER DATA 3200 POINTS LAG = 40

			SMOOTHED SPECTRUM WITH INDICATED CONFIDENCE LIMITS					
LAG	COVARIANCE SPECTRUM	RAW SPECTRUM	90 PCT. LOWER LIMIT	AVERAGE	90 PCT. UPPER LIMIT	95 PCT. LOWER LIMIT	AVERAGE	95 PCT. UPPER LIMIT
0	0.0000	0.6538	0.6482	0.7171	0.7716	0.6210	0.7171	0.7773
1	0.0000	0.7803	0.6429	0.7112	0.7652	0.6159	0.7112	0.7709
2	0.0000	0.6303	0.5622	0.6219	0.6691	0.5385	0.6219	0.6741
3	0.0000	0.4465	0.4584	0.5071	0.5457	0.4392	0.5071	0.5497
4	0.0000	0.5051	0.4252	0.4704	0.5061	0.4073	0.4704	0.5099
5	0.0000	0.4247	0.3890	0.4303	0.4630	0.3726	0.4303	0.4664
6	0.0000	0.3666	0.3270	0.3617	0.3892	0.3132	0.3617	0.3921
7	0.0000	0.2889	0.2643	0.2923	0.3146	0.2532	0.2923	0.3169
8	0.0000	0.2249	0.2055	0.2273	0.2446	0.1968	0.2273	0.2464
9	0.0000	0.1705	0.1676	0.1853	0.1994	0.1605	0.1853	0.2009
10	0.0000	0.1755	0.1443	0.1596	0.1718	0.1382	0.1596	0.1730
11	0.0000	0.1170	0.1215	0.1344	0.1446	0.1164	0.1344	0.1457
12	0.0000	0.1281	0.1061	0.1174	0.1263	0.1016	0.1174	0.1272
13	0.0000	0.0962	0.0889	0.0984	0.1058	0.0852	0.0984	0.1066
14	0.0000	0.0729	0.0741	0.0819	0.0882	0.0710	0.0819	0.0888
15	0.0000	0.0957	0.0691	0.0764	0.0823	0.0662	0.0764	0.0829
16	0.0000	0.0614	0.0635	0.0703	0.0756	0.0608	0.0703	0.0762
17	0.0000	0.0725	0.0604	0.0669	0.0719	0.0579	0.0669	0.0725
18	0.0000	0.0611	0.0565	0.0625	0.0673	0.0542	0.0625	0.0678
19	0.0000	0.0556	0.0499	0.0551	0.0593	0.0478	0.0551	0.0598
20	0.0000	0.0484	0.0433	0.0479	0.0516	0.0415	0.0479	0.0520
21	0.0000	0.0394	0.0382	0.0423	0.0455	0.0366	0.0423	0.0458
22	0.0000	0.0419	0.0380	0.0420	0.0452	0.0364	0.0420	0.0455
23	0.0000	0.0448	0.0370	0.0410	0.0441	0.0355	0.0410	0.0444
24	0.0000	0.0323	0.0323	0.0358	0.0385	0.0310	0.0358	0.0388
25	0.0000	0.0336	0.0307	0.0339	0.0365	0.0294	0.0339	0.0368
26	0.0000	0.0362	0.0313	0.0347	0.0373	0.0300	0.0347	0.0376
27	0.0000	0.0327	0.0293	0.0325	0.0349	0.0281	0.0325	0.0352
28	0.0000	0.0282	0.0255	0.0282	0.0303	0.0244	0.0282	0.0305
29	0.0000	0.0235	0.0227	0.0251	0.0270	0.0217	0.0251	0.0272
30	0.0000	0.0251	0.0230	0.0254	0.0273	0.0220	0.0254	0.0275
31	0.0000	0.0279	0.0243	0.0269	0.0289	0.0233	0.0269	0.0291
32	0.0000	0.0267	0.0245	0.0271	0.0292	0.0235	0.0271	0.0294
33	0.0000	0.0273	0.0243	0.0268	0.0289	0.0232	0.0268	0.0291
34	0.0000	0.0262	0.0236	0.0261	0.0281	0.0226	0.0261	0.0283
35	0.0000	0.0248	0.0215	0.0238	0.0256	0.0206	0.0238	0.0258
36	0.0000	0.0193	0.0192	0.0212	0.0228	0.0184	0.0212	0.0230
37	0.0000	0.0213	0.0205	0.0226	0.0244	0.0196	0.0226	0.0246
38	0.0000	0.0286	0.0244	0.0270	0.0291	0.0234	0.0270	0.0293
39	0.0000	0.0296	0.0235	0.0260	0.0280	0.0226	0.0260	0.0282
40	0.0000	0.0163	0.0208	0.0230	0.0247	0.0199	0.0230	0.0249

CHAPTER VI

CONCLUSIONS AND RECOMMENDATIONS

In this thesis, an attempt has been made to provide effective methods of statistically analyzing porous media. It has been shown that the properties derived from communication theory, the autocovariance estimate and the power spectrum, are capable of differentiating between different media. Further, it would appear from the work with the simple representations of media which were created for this problem, that real porous media lends itself to this type of analysis.

Perhaps the most significant conclusion that can be drawn from this thesis then, is that the techniques of statistical communication theory are adaptable for this form of analysis. Although the classical Fourier approach did not provide consistent results, the Tukey Hanning [63] power spectrum estimate offered good discrimination between the different media.

There are several obvious steps required to further the proof and development of the theory presented herein. A large number of samples from each simple type of media should be analyzed to affirm the consistency of results. Then new samples should be fabricated with varying proportions of the

different size beads, to determine how fine a discrimination can be made.

If these tools continue to offer good results with idealized media, the next step would be to test them on real rock samples. Such statistical methods as Discrimination Function Analysis or Factor Analysis might be applied to the data to improve the differentiation between "different" media.

Finally, if these techniques prove successful upon real media, it would be logical to then attempt to relate the statistical properties, such as the power spectrum, to the static and dynamic fluid behavior of the media.

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APPENDIXES

APPENDIX A

SINGLE FOURIER SERIES

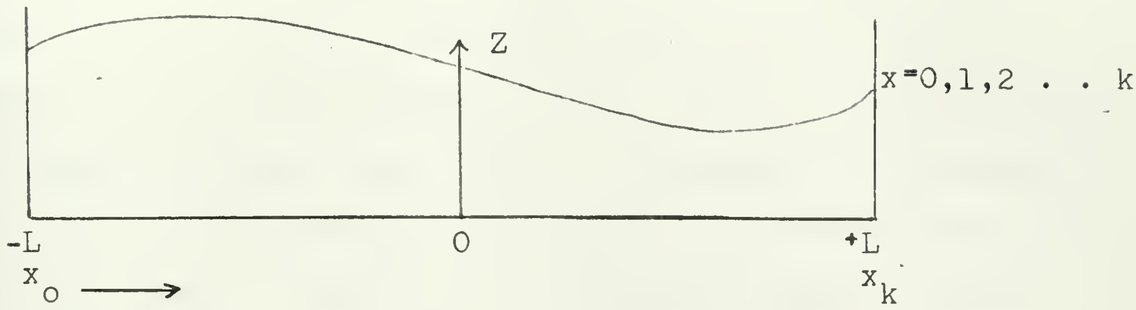
In many technical problems, it is necessary to deal with complicated periodic functions, that is functions $F(x) = F(x + T)$ for all values of x, T being the period of the function. Use of the Fourier series allows us to write these complex functions in terms of simple trignometric functions. Although Fourier series applications are not restricted to periodic functions, as explained in Gaskell,[64] special care must be taken in applying the techniques, the discussion of which is beyond the scope of this paper. Most texts on the use of the Fourier series provide the necessary instruction in this area, e.g., Tolstov.[65]

Although the complete theory of Fourier series expansions are covered in detail in such reference material as Tolstov, Gaskell,[66] Hildebrand,[67] and others, a resumé of the simple derivation for a periodic orthogonal function is included herein to assist the reader in the basic understanding of the uses of the harmonic analysis that follow.

If we have a function, such that there is one independent variable and one dependent variable, taken between the limits $-L < O < L$, we could write the function:

$$z = F(x) \quad \text{or;} \quad z = \int_{-L}^{+L} x dx$$

We might illustrate the same function:



The data set, (x_i, z_i) is such that $x_o = -L$, $(x_{i+1} - x_i)$ is constant and $x_k = +L$, the value of "k" being even.

We then assume that an equation using simple sine and cosine terms may be written, which will represent the function, as follows:

$$Z = f(x) = \frac{a_o}{2} + a_1 \cos \frac{\pi x}{L} + a_2 \cos \frac{2\pi x}{L} + \ldots a_n \cos \frac{n\pi x}{L}$$

$$+ b_1 \sin \frac{\pi x}{L} + b_2 \sin \frac{2\pi x}{L} + \ldots b_n \sin \frac{n\pi x}{L}$$

or;

$$Z = f(x) = \frac{a_o}{2} + \sum_{n=1}^{\infty} (a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L}) \qquad (1)$$

The degree to which this equation, so written, will converge upon the function, may be shown to be dependent upon two factors. The first factor is the number of harmonics (n) to which the series is expanded, and the second is the number of data points observed in relation to the basic length. The author has achieved an accurate fit to the degree of .974 on

an eighty point function of length 2π, with data values ranging from -1 to +1. The series was expanded to forty harmonics. By holding either of the stated factors constant, and decreasing the other, a loss in accuracy of representation was realized. The reason for the "2" in the denominator of the first term of equation (1) will become apparent as we find the coefficients a_n and b_n of the series. By multiplying each side of the above equation by cos nx, and integrating between the limits of +L and -L, Gaskell shows that the equation may be rewritten:

$$\int_{-L}^{+L} f(x)\cos\frac{n\pi x}{L}\,dx = a_n \int_{-L}^{+L} \cos^2\frac{n\pi x}{L}\,dx = La_n$$

or:

$$a_n = \frac{1}{L}\int_{-L}^{+L} f(x)\cos\frac{n\pi x}{L}\,dx \tag{2}$$

With the a_o term as the first term of the equation (1), utilizing the same steps, we discover:

$$\int_{-L}^{+L} f(x)\cos\frac{(o)\pi x}{L}\,dx = \int_{-L}^{+L} f(x)\,dx = \int_{-L}^{+L} \frac{1}{2}a_o\,dx = La_o$$

$$a_o = \frac{1}{L}\int_{-L}^{+L} f(x)\,dx = \frac{1}{L}\int_{-L}^{+L} f(x)\cos\frac{(o)\pi x}{L}\,dx \tag{3}$$

Gaskell further notes that $\frac{a_o}{2}$ is the average value of f(x) over the entire interval.[68] Using the trapezoidal rule to numerically integrate Equation (2) over its limits, and

utilizing equally spaced points ($\Delta x = \frac{2L}{k}$) where x = 1, 2, 3, . . . k, we can approximate Equation (2) as follows:

$$a_n = \frac{1}{L} \cdot \Delta x \left[\frac{z_o + z_k \cos^{n\pi}}{2} + \sum_{i=1}^{i=k-1} z_i \cos \frac{n\pi x_i}{L} \right] =$$

$$\frac{2}{k} \left[\frac{z_o + z_k \cos^{n\pi}}{2} + \sum_{i=1}^{i=k-1} z_i \cos \frac{n\pi x_i}{L} \right] \tag{4}$$

It is shown by Whittaker and Robinson,[69] that use of the trapezoidal rule produces coefficients which satisfy the least squares criterion of a "best" fit to the existing data, that is,

$$\sum_{i=o}^{i=k-1} (z_i - Px_i)^2 = \text{Minimum} \tag{5}$$

where Px_i is the polynomial approximation to be observed z_i.

The same reasoning as above can be enlisted to determine the coefficient b_n, except that the sine terms are zero at the end points, (sine (0) = 0) and b_n becomes:

$$b_n = \frac{2}{k} \sum_{i=1}^{i=k-1} z_i \sin \frac{n\pi x_i}{L} \tag{6}$$

Now, assuming we have discrete data z_i for each value of x_i taken at equally spaced points between the limits, we can now compute the values of the coefficients, and having so done, substitute back into Equation (1), which becomes a one-equation representation of the function. The coefficients, a_n and b_n, serve to characterize, in one way, the function.

However, this means of characterization has the distinct disadvantage of being dependent upon the phase angle of the sinusoids into which the function is resolved.[70] This disadvantage may be overcome in several ways. Some of the more complex characterization techniques, which we shall discuss and compare in latter sections, are the use of the autocorrelation function, and the power spectrum. These functions are non-dependent of phase angle, as we will later show. A less complex technique, which still dismisses the dependency upon the phase angle is discussed in a paper on wind generated sea waves, distributed by New York University.[71] An estimate of the power spectrum is made in the following manner, using the Fourier coefficients:

$$\text{Estimated Power Spectrum} \quad c_n^2 = a_n^2 + b_n^2 \qquad (7)$$

Although this estimate is a characterization of the function, and does dismiss the dependency upon the phase angle, the same reference notes that the quantity c_n^2 is a very unstable estimate of the energy at any frequency. Only through a weighting process of the c_n values, would an accurate spectrum, such as that determined by the Tukey method[72] be recovered. Lee[73] uses a similar estimate of the power spectrum, with the exception that he divides the above estimate by a factor of four.

$$c_n^2 = \frac{a_n^2 + b_n^2}{4} \qquad (8)$$

In that the basis for our discussion dealing with auto-correlation and power spectrum analysis evolves from this same reference, and since the division does not affect the term for our purposes, we have used the "Lee" estimate in our work. This usage will allow a means of comparing this simple estimate of the power spectrum with the values computed by more complex techniques.

APPENDIX B

DOUBLE FOURIER SERIES

As in the case of a function with one independent variable discussed in an earlier section, it is also possible to represent with a Fourier series, a dependent variable as the function of two independent variables.

$$z = f(x,y) \tag{1}$$

A rigorous discussion regarding orthogonal systems in two variables may be found in Tolstov.[74]

Thus, a parameter such as z may be considered to be oscillatory in two mutually perpendicular directions, such as east-west and north-south, allowing Equation (1) to represent the areal variation in z. If we consider a function z to have a fundamental period of 2L along the x coordinate, and 2H along the y axis, then following the same reasoning discussed earlier in the single dimension case, we can write Double Fourier series function:

$$z \cong P_{x,y} = \sum_{m=o}^{m=M} \sum_{n=o}^{n=N} \lambda_{m,n} \Big[a_{m,n} \cos \frac{\pi m x}{L} \cos \frac{\pi n y}{H} + b_{m,n} \sin \frac{\pi m x}{L} \cos \frac{\pi n y}{H} + c_{m,n} \cos \frac{\pi m x}{L} \sin \frac{\pi n y}{H} + d_{m,n} \sin \frac{\pi m x}{L} \sin \frac{\pi n y}{H} \Big] \tag{2}$$

where

$\lambda_{m,n} = 1/4$ $\quad m = n = o$

$\lambda_{m,n} = 1/2$ $\quad m = o, n>o$ or $m>o, n=o$

$\lambda_{m,n} = 1$ $\quad m>o, n>o$

M,N number of harmonics considered (Should be equal to or less than one half the number of data points considered in the x and y directions respectively.)[75]

The reasoning behind the quarter and half terms follows also from the one dimension analysis, in that the end terms considered in a numerical integration are averaged, according to the theory of the trapezoidal rule.[76] In the two dimension case, the averaging process must be carried in each direction, thus the quarter terms. Again, the problem is to evaluate the coefficients. Integrating Equation (2), the coefficients are determinable from the following relations:

$$a_{m,n} = \frac{1}{LH} \int_{-H}^{+H} \int_{-L}^{+L} f(x,y) \cos \frac{\pi m x}{L} \cos \frac{\pi n y}{H} \, dydx \qquad (3)$$

$$b_{m,n} = \frac{1}{LH} \int_{-H}^{+H} \int_{-L}^{+L} f(x,y) \sin \frac{\pi m x}{L} \cos \frac{\pi n y}{H} \, dydx \qquad (4)$$

$$c_{m,n} = \frac{1}{LH} \int_{-H}^{+H} \int_{-L}^{+L} f(x,y) \cos \frac{\pi m x}{L} \sin \frac{\pi n y}{H} \, dydx \qquad (5)$$

$$d_{m,n} = \frac{1}{LH} \int_{-H}^{+H} \int_{-L}^{+L} f(x,y) \sin \frac{\pi m x}{L} \sin \frac{\pi n y}{H} \, dydx \qquad (6)$$

It is possible to numerically integrate Equations 3-6 utilizing the trapezoidal rule. Again it must be remembered that the integration must be carried out in two dimensions, and special care must be taken with regard to the corner points. In order to clarify the following derivation, Figure 1 displays the nomenclature to be used.

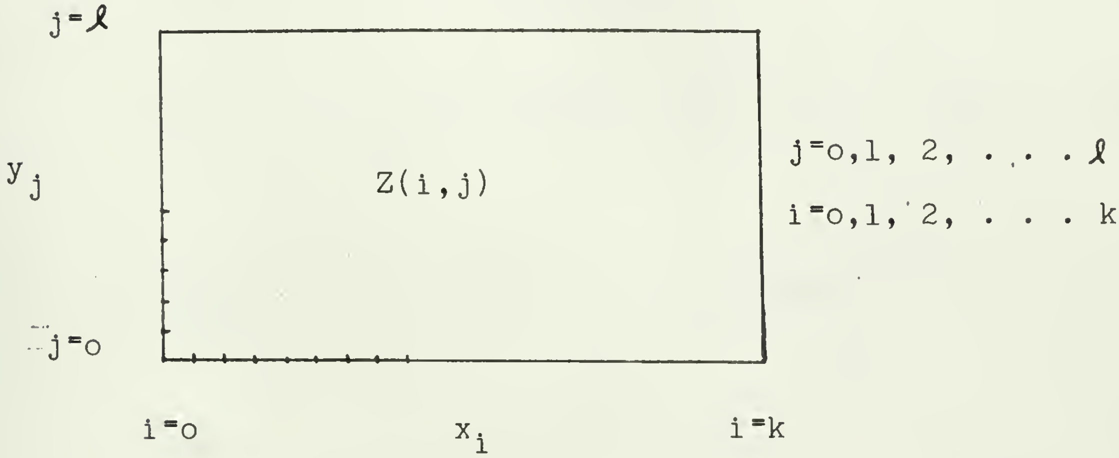


Figure 1

According to the proper use of the trapezoidal rule, it can be shown that for the one dimension case (one independent variable)

$$\int_{-L}^{+L} z(x)\, dx \cong \Delta x \left[\frac{z_o + z_k}{2} + \sum_{i=1}^{i=k-1} z_i \right] \qquad (7)$$

where $\Delta x = \frac{2L}{k}$

Considering the second dimension of depth in our picture, when we are concerned with but one direction, we are able to write Equation (7)

$$\int_{-L}^{+L} z(x)\,dx \cong \Delta x \left[\frac{z_{o,j} + z_{k,j}}{2} + \sum_{i=1}^{i=k-1} z_{i,j} \right] \qquad (8)$$

However, in a specific case, where it is required that integration be applied in two dimensions, as will be done for Equation (3), the following steps are taken:

$$a_{m,n} = \frac{1}{L} \cdot \frac{1}{H} \int_{-H}^{+H} \int_{-L}^{+L} z(x,y)\,dxdy$$

$$\cong \frac{1}{LH} \int_{-H}^{+H} \Delta x \left[\frac{z_{o,j} + z_{k,j}}{2} + \sum_{i=1}^{i=k-1} z_{i,j} \right] dy$$

$$\cong \frac{\Delta x \Delta y}{LH} \left[\left[\frac{\frac{z_{o,o} + z_{k,o}}{2} + \frac{z_{o,\ell} + z_{k,\ell}}{2}}{2} \right]^{(A)} + \right.$$

$$\left[\sum_{j=1}^{j=\ell-1} \frac{z_{o,j} + z_{k,j}}{2} \right]^{(B)} + \left[\sum_{i=1}^{i=k-1} \frac{z_{i,o} + z_{i,\ell}}{2} \right]^{(C)} +$$

$$\left. \left[\sum_{j=1}^{j=\ell-1} \sum_{i=1}^{i=k-1} z_{i,j} \right]^{(D)} \right] \qquad (9)$$

where: (A) term that averages the corners in both directions.

(B) term that averages opposite vertical edges (less corners).

(C) Term that averages opposite horizontal edges (less corners).

(D) term that sums all interior points.

The number of points in either direction should be an odd number, requiring "k" and " " to be even numbers. Carrying the procedure to completion and reinserting the trignometric terms, which were omitted to simplify the explanation, we can write the coefficient as follows:

$$a_{m,n} = \frac{1}{\cancel{LH}} \cdot \frac{2\cancel{L}}{k} \cdot \frac{2\cancel{H}}{l} \left[\left[\frac{\frac{z_{o,o}+z_{k,o}}{2} + \frac{z_{o,}+z_{k,\ell}}{2}}{2} \cos m\pi \cos n\pi \right] (A) \right.$$

$$+ \left[\sum_{j=1}^{j=\ell-1} \frac{z_{o,j}+z_{k,j}}{2} \cos m\pi \cos \frac{n\pi y_j}{H} \right] (B)$$

$$+ \left[\sum_{i=1}^{i=k-1} \frac{z_{i,o}+z_{i,\ell}}{2} \cos \frac{m\pi x_i}{L} \cos n\pi \right] (C)$$

$$\left. + \left[\sum_{j=1}^{j=\ell-1} \sum_{i=1}^{i=k-1} z_{i,j} \cos \frac{m\pi x_i}{L} \cos \frac{n\pi y_j}{J} \right] (D) \right] \qquad (10)$$

Similarly, Equations (4-6) become:

$$b_{m,n} = \frac{4}{lk} \left[\left[\text{No (A) term in that on the corner points, } \sin \frac{m\pi x_i}{L} \longrightarrow o \right] + \right.$$

$$\left[\text{No B term due to the fact that } x_o \text{ and } x_k \text{ are equal to L and sine term goes to zero.} \right] +$$

$$\left[\sum_{i=1}^{i=k-1} \frac{z_{i,o}+z_{i,\ell}}{2} \sin \frac{m\pi x_i}{L} \cos n\pi \right] (C) +$$

$$\left. \left[\sum_{j=1}^{j=\ell-1} \sum_{i=1}^{i=k-1} z_{i,j} \sin \frac{m\pi x_i}{L} \cos \frac{n\pi y_j}{H} \right] (D) \right] \qquad (11)$$

$$c_{m,n} = \frac{4}{k}\left[\left[\text{No (A) term because sine terms go to zero.}\right]\right. +$$

$$\left[\sum_{j=1}^{j=\ -1} \frac{z_{o,j}+z_{k,j}}{2} \cos m\pi \sin \frac{n\pi y_j}{H}\right]^{(B)} +$$

$$\left[\text{No (C) term because sine terms go to zero.}\right] +$$

$$\left.\left[\sum_{j=1}^{j=\ell-1} \sum_{i=1}^{i=k-1} z_{i,j} \cos \frac{m\pi x_i}{L} \sin \frac{n\pi y_j}{H}\right]^{(D)}\right] \qquad (12)$$

$$d_{m,n} = \frac{4}{k}\left[\left[\text{No (A), (B), or (C) terms because sine terms to to zero.}\right]\right. +$$

$$\left.\left[\sum_{j=1}^{j=\ell-1} \sum_{i=1}^{i=k-1} z_{i,j} \sin \frac{m\pi x_i}{L} \sin \frac{n\pi y_j}{H}\right]^{(D)}\right] \qquad (13)$$

APPENDIX C

COMPUTER PROGRAM TO DETERMINE COEFFICIENTS FOR SINGLE FOURIER SERIES (ONE DIMENSION) FIT OF ORTHOGONAL FUNCTIONS

Purpose

The purpose of this program is to determine the Fourier coefficients a_n and b_n, when a function, f(x), is fitted by a single or one-dimensional Fourier series. The secondary purpose is to calculate an estimate of the power spectrum of the function, which in turn is a characterization of the function.

Language

Fortran IV (IBM 7040 Computer)

Symbolic Dictionary

Variable	S/A*	I/O**	Description
X	A	I/O	Independent variable data point.
Z	A	I/O	Dependent variable data point.
MZ	A	I	Fixed point value for Z.
A	A	O	Fourier coefficient.
B	A	O	Fourier coefficient.

* S--Single Variable; A--Array of Variables

** I--Input Variable; O--Output Variable

Variable	S/A	I/O	Description
C	A	O	Estimate of the function's power spectrum based upon the equation: $C_n^2 = A_n^2 + B_n^2$.
P	A	O	Estimate of input function Z at each data point, created by using calculated coefficients.
R	A	O	Residual or difference between input function Z and estimated function P, at each data point.
CC	S	--	$A(N)^2 + B(N)^2$
K	S	I	Number of data points.
EL	S	I	Half-length of sample.
KOW	S	I	Number of samples.
KOO	S	I	Sample index.
PIE	S	I	Mathematical expression input as being equal to 3.14159265.
KKK	S	I	Number of harmonics to which the Fourier series is to be expanded.
ZSUM	S	--	Sum of Z data point values.
SUMCS	S	--	Sum of Sine and Cosine terms.
SUMSIN	S	--	Sum of Sine terms in equation to determine coefficients.
SUMCOS	S	--	Sum of Cosine terms in equation to determine coefficients.
AZERO	S	O	First term of Fourier expansion.
I	S	O	Data point index.
PMAX	S	O	Maximum value of variable P.
PMIN	S	O	Minimum value of variable P.

Variable	S/A	I/O	Description
ZMAX	S	O	Maximum value of variable Z.
ZMIN	S	O	Minimum value of variable Z.
RMAX	S	O	Maximum value of variable R.
RMIN	S	O	Minimum value of variable R.
N	S	--	Harmonic index.

Program Routine

The program utilizes data points equally spaced along the function from a set origin, and by using Fortran equivalents of Equations (4) and (6) from Appendix A, generates a Fourier series estimate of the function. The Fourier coefficients generated are presented as characterizing the function, and by combining them according to the "Lee" estimate, $C(N)^2 = (A(N)^2 + B(N)^2)/4$, a characterizing power spectrum which is independent of phase angle, is determined. When more than one sample is run, the C(N) values are grouped statistically for analysis. The program checks the values of the coefficients by creating an estimate of the original function, using Equation (1) from Appendix A. A plot of the original function, the created estimate, and the residuals, or differences between the two is presented.

```
C
C               PROGRAM  -FOURIER-           AUGUST 1965
C        SINGLE FOURIER SERIES PROGRAM (ONE DIMENSION)
C        PROGRAMMED BY WD ALDENDERFER
C         ONE DIMENSION FIT OF 3MM BEADS DATA
C       400 POINTS     50 HARMONICS
C
      DIMENSION X(801),Z(801),MZ(801),A(30),B(30),C(30,30),P(801),
     1IDSH(32), TITLE(13),LO(30),LC(30),LR(30),R(801)
      CALL CLOCK  (0)
C
      READ (5,102) (TITLE(I),I=1,13)
  102 FORMAT (13A6)
      READ (5,1) K, EL
    1 FORMAT (I10,10X,F10.0)
      WRITE(6,6000)
 6000 FORMAT(1H1,20X,21HPOWER SPECTRUM VALUES)
      KOW = 16
C      KOW IS THE NUMBER OF SAMPLES BEING TESTED.
      KOO = 1
      PIE = 3.14159265
      AK = K
      MK = K-1
      AMK = MK
      KKK = 50
C      KKK IS THE NUMBER OF HARMONICS
      DELX = PIE/((AK-1.)/2.)
  555 X(1) = -(PIE)
      ZSUM = 0.0
      SUMCS = 0.0
      Z(1) = 0.0
      DO 203 III = 2,K
      Z(III) = 0.0
      X(III) = X(III-1) + DELX
  203 CONTINUE
      READ(5,204) (MZ(I),I=1,K)
  204 FORMAT(80I1/80I1/80I1/80I1/80I1/I1)
      DO 205 JJ = 1,K
      IF (MZ(JJ) - 0 )800,800,801
  800 Z(JJ) = -1.0
      GO TO 205
  801 Z(JJ) = MZ(JJ)
  205 CONTINUE
      DO 12 II = 2,MK
      ZSUM = ZSUM + Z(II)
   12 CONTINUE
      SUME =(Z(1) + Z(K))/2.
      AZERO = (2./AMK) *(ZSUM + SUME)
      DO 4 N = 1,KKK
      SUMSIN = 0.0
      SOMF = 0.0
      SUMCOS = 0.0
      FN = N
      SUM= 0.0
C     ALGEBRAIC INDEX EQUALS FORTRAN INDEX LESS ONE.
      DO 5 I = 2,MK
      SUMCOS = SUMCOS +   Z(I)*COS((FN* PIE * X(I))/EL)
```

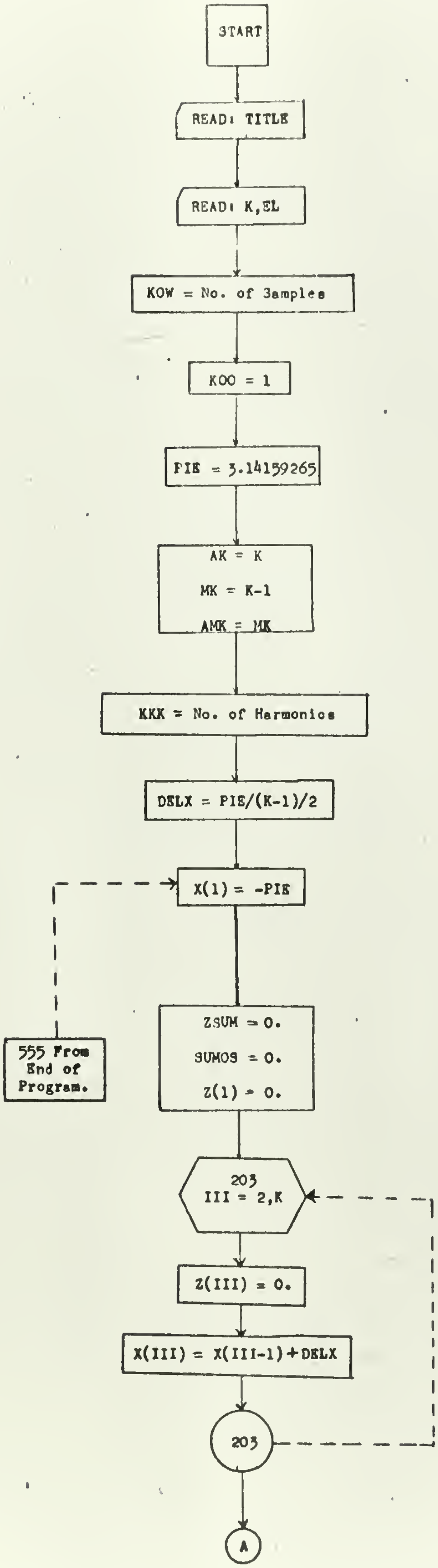
```
      SUMSIN = SUMSIN +   Z(I)*SIN((FN* PIE * X(I))/EL)
    5 CONTINUE
      SOME = ((Z(1) + Z(K))/2.0)* COS(FN*PIE)
C     ------------------------------------------------------------------
C
C      COMPUTING THE VALUE OF THE POWER SPECTRUM ESTIMATE .
C      ESTIMATE = A(N) SQUARED + B(N) SQUARED DIVIDED BY FOUR
C
C     ------------------------------------------------------------------
      A(N) = 2./AMK * (SUMCOS + SOME)
      B(N) = 2.0/AMK * SUMSIN
      CC = A(N)* A(N) + B(N) * B(N)
      C(KOO,N) = CC/4.0
    4 CONTINUE
      WRITE(6,687)(C(KOO,N),N=1,50)
  687 FORMAT((1X,10F10.4))
      WRITE(6,6005)
 6005 FORMAT(1H0,11HNEXT SAMPLE)
      GO TO 686
    2 NMAX = (K-1)/2
      DO 8 I = 1,K
      SUMCS = AZERO / 2.
      DO 9 N=1,KKK
      FN = N
      SUMCS = SUMCS + A(N) * COS((FN* PIE * X(I))/EL) +
     1B(N) * SIN((FN * PIE * X(I))/EL)
    9 CONTINUE
      P(I) = SUMCS
      R(I) = P(I) - Z(I)
    8 CONTINUE
      GO TO 213
C     ------------------------------------------------------------------
C
C                          PLOTTING ROUTINE
C
C     ------------------------------------------------------------------
  214 ISTAR = -13702990896
      IEYE  =  27661634608
      IBLANK = -17997958192
      IDASH  = -818089008
      DO 10 I = 1,32
   10 IDSH(I)= IDASH
      WRITE (6,103)(TITLE(I),I=1,13)
  103 FORMAT (1H1,13A6//,17X,8HOBSERVED,32X,10HCALCULATED,
     132X,8HRESIDUAL)
      WRITE (6,104) (IDSH(I),I=1,32),(IDSH(I),I=1,32),
     1(IDSH(I),I=1,32)
  104 FORMAT(5X,32A1,9X,32A1,9X,32A1)
      ZMAX = Z(1)
      ZMIN = Z(1)
      PMAX = P(1)
      PMIN = P(1)
      RMAX = R(1)
      RMIN = R(1)
      DO 20 I = 2, K
      IF (Z(I).GT. ZMAX) ZMAX = Z(I)
      IF (Z(I).LT. ZMIN) ZMIN = Z(I)
```

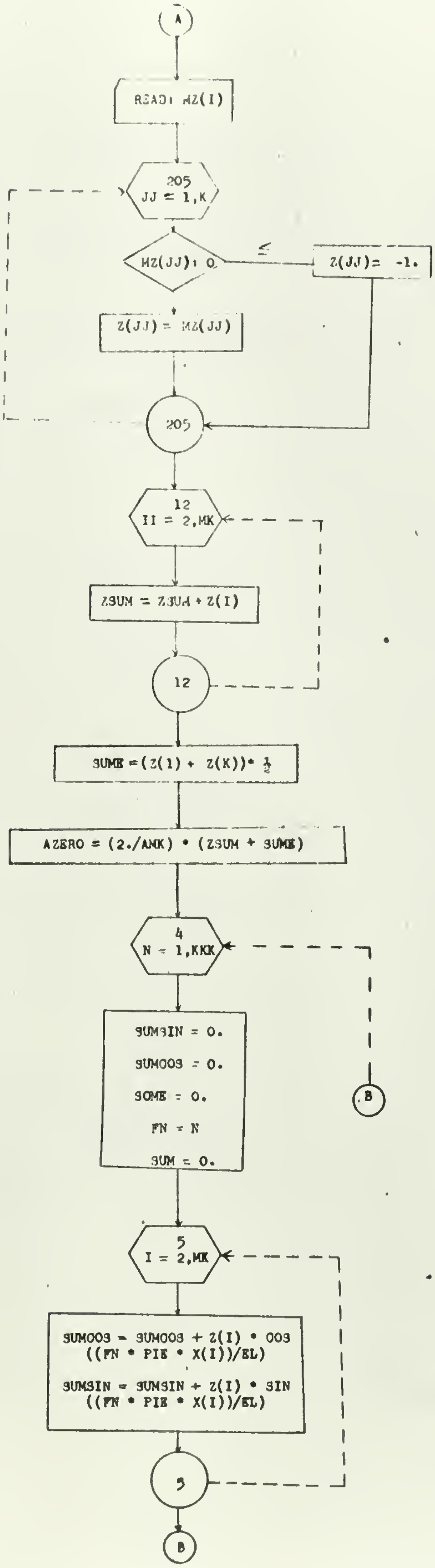
```
      IF (P(I).GT. PMAX) PMAX = P(I)
      IF (P(I).LT. PMIN) PMIN = P(I)
      IF (R(I).GT. RMAX) RMAX = R(I)
   20 IF (R(I).LT. RMIN) RMIN = R(I)
      WRITE(6,902)  ZMAX,ZMIN,PMAX,PMIN,RMAX,RMIN
  902 FORMAT(1X,4HCHE2,6F10.4)
      DO 250 I = 1,30
      LO(I) = IBLANK
      LC(I) = IBLANK
  250 LR(I) = IBLANK
      KHAF =(K+1)/2
      DO 30 I= 1,K
      JZ =(((Z(I)-ZMIN)/(ZMAX-ZMIN))* 29.) + 1.
      JO =(((P(I)-PMIN)/(PMAX-PMIN))* 29.) + 1.
      JR =(((R(I)-RMIN)/(RMAX-RMIN))* 29.) + 1.
      LO (JZ) =ISTAR
      LC (JO) =ISTAR
      LR (JR) =ISTAR
      KK = I - KHAF
      WRITE(6,105) KK,(LO(JJ),JJ=1,30), KK,(LC(JJ),JJ=1,30),
     1KK,(LR(JJ),JJ=1,30)
  105 FORMAT (1X,I4, 1HI, 30A1,1HI, 5X,I4,1HI,30A1,1HI,5X,
     1I4,1HI,30A1,1HI    )
      LO(JZ)= IBLANK
      LC(JO)= IBLANK
      LR(JR)= IBLANK
   30 CONTINUE
C     ----------------------------------------------------------------
C                         END OF PLOTTING ROUTINE
C     ----------------------------------------------------------------
  213 DO 699 I = 1,K
      WRITE(6,698)  I,Z(I), P(I),R(I)
  698 FORMAT(10X,I5,3F20.8)
  699 CONTINUE
  686 KOO = KOO + 1
      IF(KOO.GT.KOW)  GO TO 400
      GO TO 555
  400 CALL CLOCK(1)
C     ----------------------------------------------------------------
C
C      GROUPING OF C(N) FACTORS (POWER SPECTRUM) FOR FREQUENCY
C      STUDY. POWER SPECTRUM VALUE = A*A + B*B DIVIDED BY
C      4.0 AS DEFINED IN THE TEXT BY LEE. THIS IS ONLY AN
C      ESTIMATE OF THE TRUE SPECTRUM.
C
C     ----------------------------------------------------------------
      WRITE(6,577)
  577 FORMAT(1H1,37HFREQUENCY DISTRIBUTION OF C(N) VALUES)
      WRITE(6,575)
  575 FORMAT(1H0,3X,1HN,6X,4HINT1,6X,4HINT2,6X,4HINT3,6X,
     14HINT4,6X,4HINT5,6X,4HINT6,6X,4HINT7,6X,4HINT8,6X,4HINT9,
     26X,5HINT10,5X,5HINT11,5X,5HINT12)
      WRITE(6,576)
  576 FORMAT(1H0,9X,40HLT .001    .001-.01   .01-.02    .02-.03    ,
     150H.03-.04    .04-.05    .05-.06    .06-.07    .07-.08    .,
     225H08-.09    .09-.1      GT.1)
```

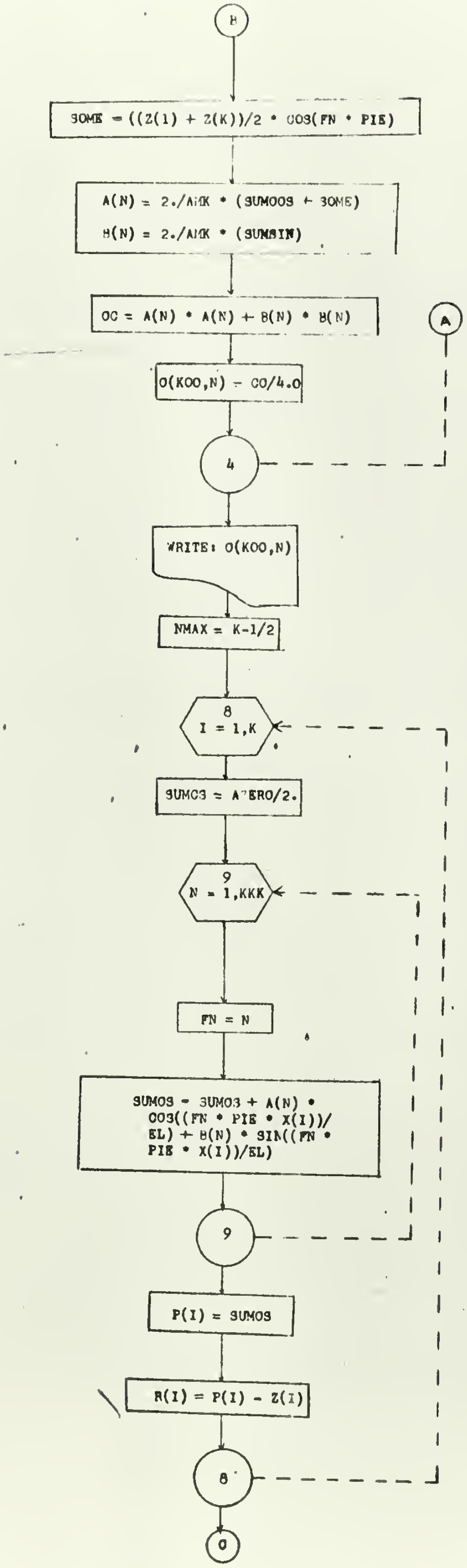
```
      DO 500 N = 1,KKK
      TOT = 0.0
      TOTS = 0.0
      INT1 = 0
      INT2 = 0
      INT3 = 0
      INT4 = 0
      INT5 = 0
      INT6 = 0
      INT7 = 0
      INT8 = 0
      INT9 = 0
      INT10 = 0
      INT11 = 0
      INT12 = 0
      DO 501 KOO = 1,KOW
      COEFF = C(KOO,N)
      TOT = TOT + COEFF
      TOTS = TOTS + (COEFF*COEFF)
      IF(COEFF.GT..00099995)  GO TO 505
      INT1 = INT1 + 1
      GO TO 501
  505 IF(COEFF.GT..00995)  GO TO 506
      INT2 = INT2 + 1
      GO TO 501
  506 IF(COEFF.GT..019995)  GO TO 507
      INT3 = INT3 + 1
      GO TO 501
  507 IF(COEFF.GT..029995)  GO TO 508
      INT4 = INT4 + 1
      GO TO 501
  508 IF(COEFF.GT..039995)  GO TO 509
      INT5 = INT5 + 1
      GO TO 501
  509 IF(COEFF.GT..049995)  GO TO 510
      INT6 = INT6 + 1
      GO TO 501
  510 IF(COEFF.GT..059995) GO TO 511
      INT7 = INT7 + 1
      GO TO 501
  511 IF(COEFF.GT..069995) GO TO 512
      INT 8 = INT8 + 1
      GO TO 501
  512 IF(COEFF.GT..079995) GO TO 513
      INT9 = INT9 + 1
      GO TO 501
  513 IF(COEFF.GT..089995) GO TO 514
      INT10 = INT10 + 1
      GO TO 501
  514 IF(COEFF.GT..1)  GO TO 515
      INT11 = INT11 + 1
      GO TO 501
  515 INT12 = INT12 + 1
  501 CONTINUE
      AKOW = KOW
      AVE = TOT/AKOW
```

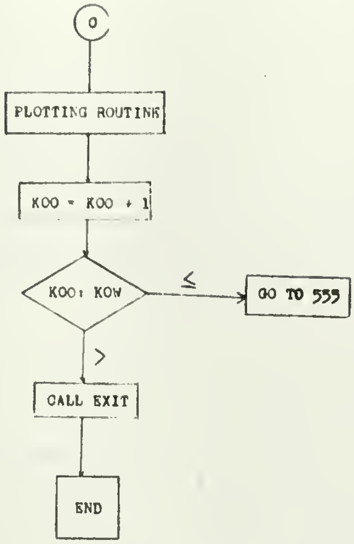
```
      AVE2 = TOTS/AKOW
      AVE3 = AVE*AVE
      SDEV = SQRT((AVE2-AVE3))
      WRITE(6,502) N,INT1,INT2,INT3,INT4,INT5,INT6,INT7,INT8,
     1INT9,INT10,INT11,INT12
  502 FORMAT(1H0,I4,12I10)
      WRITE(6,602)      AVE, SDEV
  602 FORMAT(1H0,7HMEAN = ,F20.8,10X,16HSTANDARD DEV. = ,F20.8)
C     ------------------------------------------------------------------
C                END OF GROUPING ROUTINE
C     ------------------------------------------------------------------
  500 CONTINUE
      CALL CLOCK(2)
      CALL EXIT
      END
$ENTRY
```

SINGLE FOURIER SERIES PROGRAM

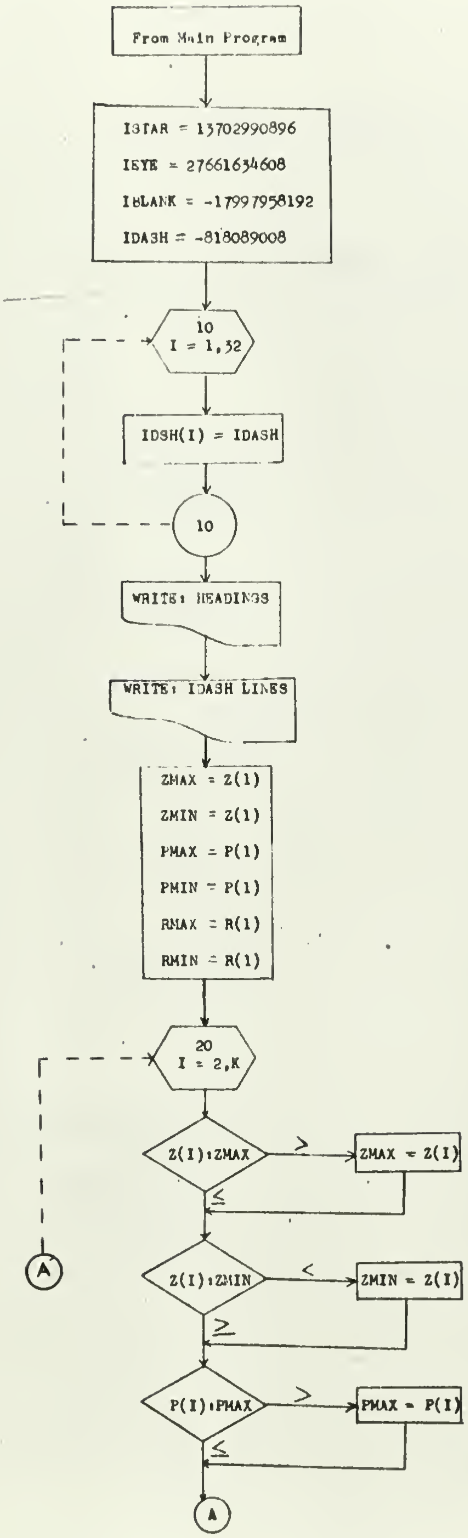


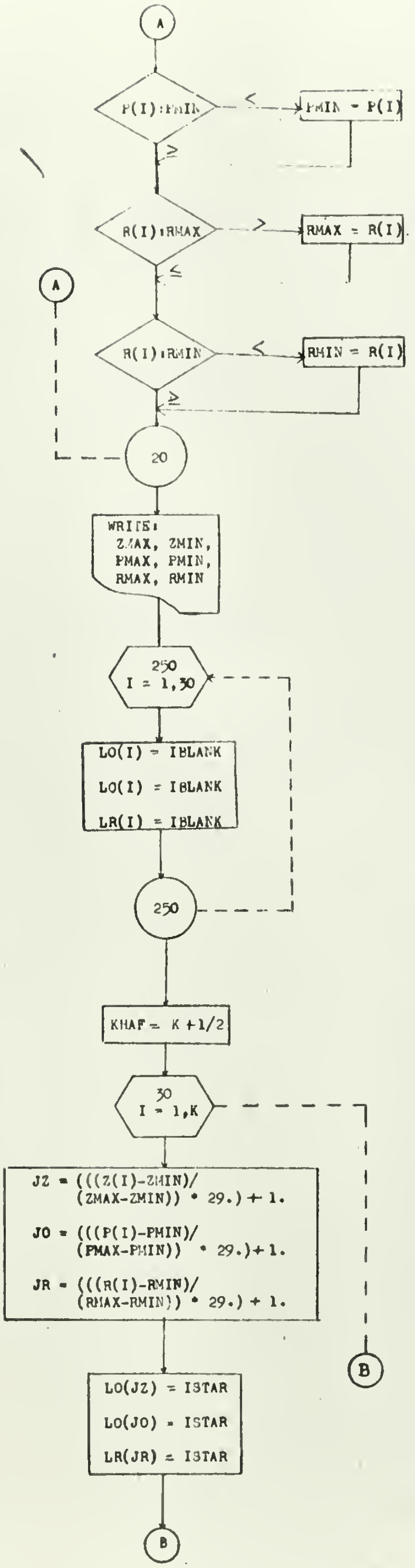

A
READ: MZ(I)
205
JJ = 1,K
MZ(JJ): O
≤
Z(JJ) = -1.
Z(JJ) = MZ(JJ)
205
12
II = 2,MK
ZSUM = ZSUM + Z(I)
12
SUME = (Z(1) + Z(K)) * ½
AZERO = (2./AMK) * (ZSUM + SUME)
4
N = 1,KKK
SUMSIN = O.
SUMCOS = O.
SOME = O.
FN = N
SUM = O.
B
5
I = 2,MK
SUMCOS = SUMCOS + Z(I) * COS ((FN * PIE * X(I))/EL)
SUMSIN = SUMSIN + Z(I) * SIN ((FN * PIE * X(I))/EL)
5
B

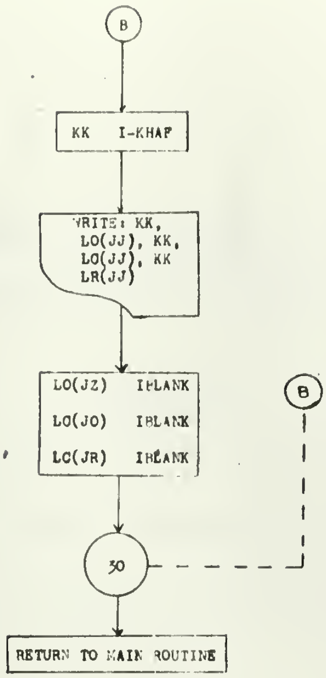

B
SOME = ((Z(1) + Z(K))/2 * COS(FN * PIE)
A(N) = 2./AMK * (SUMCOS + SOME)
B(N) = 2./AMK * (SUMSIN)
CC = A(N) * A(N) + B(N) * B(N)
A
C(KOO,N) = CC/4.0
4
WRITE: C(KOO,N)
NMAX = K-1/2
8
I = 1,K
SUMCS = AZERO/2.
9
N = 1,KKK
FN = N
SUMOS = SUMOS + A(N) *
COS((FN * PIE * X(I))/
EL) + B(N) * SIN((FN *
PIE * X(I))/EL)
9
P(I) = SUMOS
R(I) = P(I) - Z(I)
8
C


O
PLOTTING ROUTINE
KOO = KOO + 1
KOO: KOW
≤
GO TO 555
>
CALL EXIT
END

SINGLE FOURIER SERIES PLOTTING ROUTINE




A
P(I):PMIN
<
PMIN = P(I)
≥
R(I):RMAX
>
RMAX = R(I)
≤
A
R(I):RMIN
<
RMIN = R(I)
≥
20
WRITE:
ZMAX, ZMIN,
PMAX, PMIN,
RMAX, RMIN
250
I = 1,30
LO(I) = IBLANK
LO(I) = IBLANK
LR(I) = IBLANK
250
KHAF = K +1/2
30
I = 1,K
JZ = (((Z(I)-ZMIN)/
(ZMAX-ZMIN)) * 29.) + 1.
JO = (((P(I)-PMIN)/
(PMAX-PMIN)) * 29.) + 1.
JR = (((R(I)-RMIN)/
(RMAX-RMIN)) * 29.) + 1.
B
LO(JZ) = ISTAR
LO(JO) = ISTAR
LR(JR) = ISTAR
B


B
KK I-KHAF
WRITE: KK, LO(JJ), KK, LO(JJ), KK LR(JJ)
LO(JZ) IBLANK
LO(JO) IBLANK
LC(JR) IBLANK
B
30
RETURN TO MAIN ROUTINE

COMPUTER PROGRAM TO DETERMINE COEFFICIENTS FOR DOUBLE FOURIER SERIES (TWO DIMENSION) APPROXIMATION OF A SURFACE AREA

Purpose

The purpose of this program is to approximate a surface area, f(x,y), using a double Fourier series, and to present the Fourier coefficients which are a characterization of the surface.

Language

Fortran IV (IBM 7040 Computer)

Symbolic Dictionary

Variable	S/A*	I/O**	Description
X	A	I	Independent variable data point along x-axis.
Y	A	I	Independent variable data point along y-axis.
Z	A	I	Dependent variable data point.
EL	S	I	One-half of fundamental period along the x-axis.
H	S	I	One-half of fundamental period along the y-axis.
KK	S	I	Number of data points in x-direction.

---

* S--Single Variable; A--Array of Variables

** I--Input Variable; O--Output Variable

Variable	S/A	I/O	Description
LL	S	I	Number of data points in y-direction.
MMAX and NMAX	S	I	Maximum number of harmonics to which Fourier series is to be expanded. Should equal one another.
M	S	O	Harmonic index.
N	S	O	Harmonic index.
PIE	S	I	Mathematical expression input as being equal to 3.14159265.
A	A	O	Fourier coefficient.
B	A	O	Fourier coefficient.
C	A	O	Fourier coefficient.
D	A	O	Fourier coefficient.

Program Routine

This program utilizes data points equally spaced upon a grid, placed upon the surface to be analyzed. An approximation of the surface is created by expanding a double Fourier series of the type discussed in Appendix B. The Fourier coefficients are presented as characterizations of the surface.

Input Generator

Immediately following the listing of the double Fourier series program, is a listing of a simple data generator for the program. This program uses a random set of coefficients, and Equation (2) of Appendix B, to create, in the form of

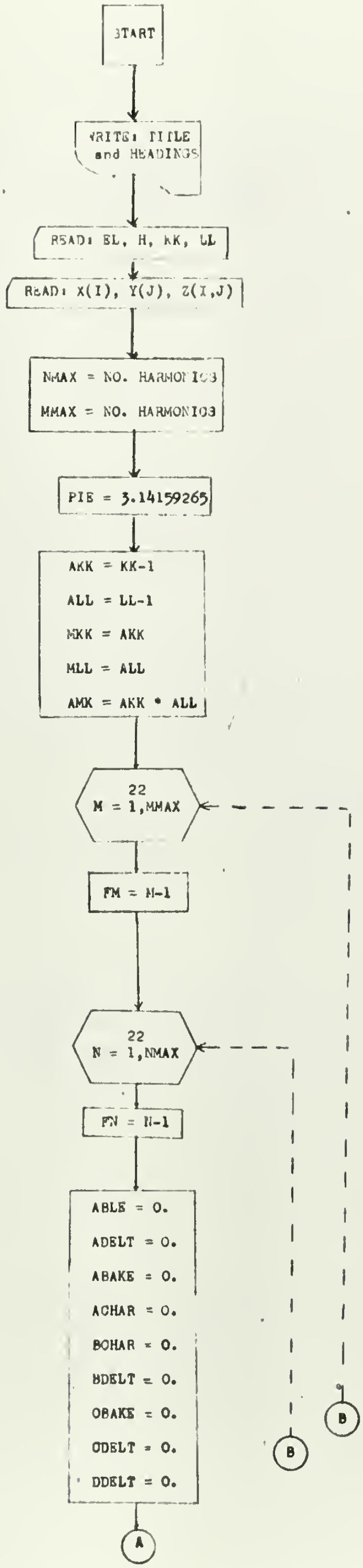
punched output, data points simulating a surface area, which are the appropriate input for the double Fourier program.

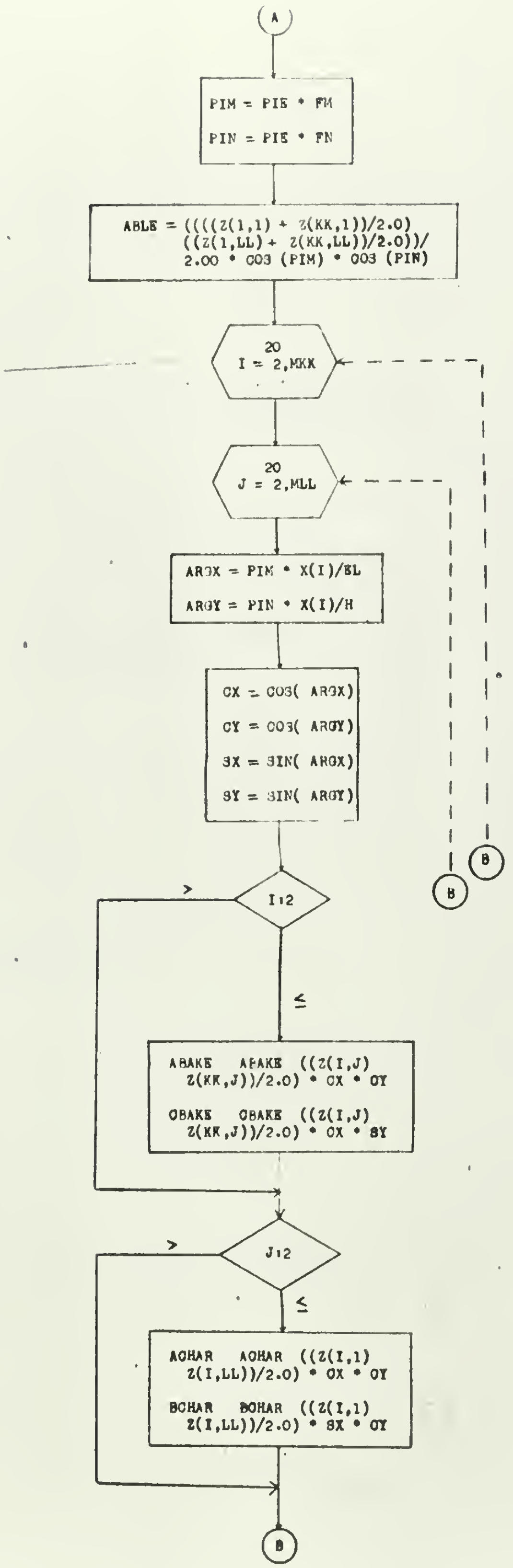
```
C     ------------------------------------------------------------------
C
C             PROGRAM  2D FOURI            AUGUST 1965
C        DOUBLE FOURIER SERIES (TWO DIMENSION)   PROGRAM
C        PROGRAMMED BY WD ALDENDERFER
C
C     ------------------------------------------------------------------
      DIMENSION X(25),Y(31),Z(25,31),A(5,5)
      DIMENSION B(5,5),C(5,5),D(5,5)
      CALL CLOCK(0)
      WRITE(6,600)
  600 FORMAT(1H1,19H2D FOURIER ANALYSIS)
      WRITE(6,623)
  623 FORMAT(1H0,20X,26HDISPLAY OF CALCULATED 2-D ,
     140HCOEFFICIENTS WHICH CHARACTERIZE FUNCTION)
      WRITE(6,654)
  654 FORMAT(1H0,4X,1HM,4X,1HN,7X,6HA(M,N),14X,6HB(M,N),
     114X,6HC(M,N),14X,6HD(M,N))
      READ(5,2)  EL,H,KK,LL
    2 FORMAT(2F20.8, 2I5)
C       EL = FUNDAMENTAL LENGTH IN THE X DIRECTION
C       H  = FUNDAMENTAL LENGTH IN THE Y DIRECTION
C       KK = NO. OF DATA POINTS IN X DIRECTION.
C       LL = NO. OF DATA POINTS IN Y DIRECTION.
      READ(5,1) ((X(I),Y(J),Z(I,J), J=1,LL),I=1,KK)
    1 FORMAT(3F20.8)
C       X = INDEPENDENT VARIABLE.
C       Y = INDEPENDENT VARIABLE.
C       Z = DEPENDENT VARIABLE.
      MMAX = 5
      NMAX = 5
C     NMAX AND MMAX ARE THE NUMBER OF HARMONICS TO WHICH THE
C     SERIES IS TO BE EXPANDED.
      PIE = 3.14159265
      AKK = KK-1
      ALL = LL-1
      MKK = AKK
      MLL = ALL
      AMK = AKK*ALL
      DO 22 M = 1,MMAX
      FM = M-1
      DO 22 N = 1,NMAX
      FN = N-1
      ABLE = 0.
      ABAKE = 0.
      ACHAR = 0.0
      ADELT = 0.0
      BCHAR = 0.0
      BDELT = 0.0
      CBAKE = 0.0
      CDELT = 0.0
      DDELT = 0.0
      PIM = PIE * FM
      PIN = PIE*FN
```

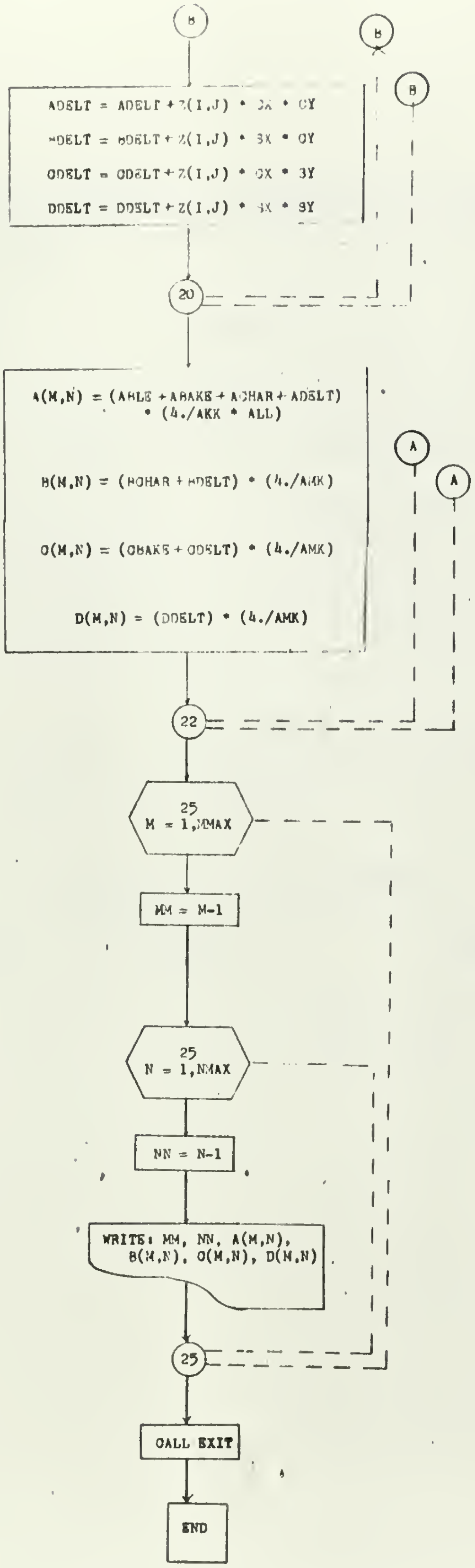
```
C
C     CALCULATE A(0,0)
C
      ABLE = ((((Z(1,1) + Z(KK,1))/2.0) + ((Z(1,LL) + Z(KK,LL))/2.0))
     1/2.0) * COS(PIM) * COS(PIN)
      DO 20 I = 2,MKK
      DO 20 J = 2,MLL
      ARGX = PIM * (X(I)/EL)
      ARGY = PIN * (Y(J)/H)
      CX = COS(ARGX)
      CY = COS(ARGY)
      SX = SIN(ARGX)
      SY = SIN(ARGY)
      IF(I.GT.2)  GO TO 400
C
C     CALCULATE COEFFICIENTS BY PARTS.
C
      ABAKE = ABAKE +((Z(1,J) + Z(KK,J))/2.0) *CX *CY
      CBAKE = CBAKE + ((Z(1,J) + Z(KK,J))/2.0) * CX * SY
  400 IF(J.GT.2)  GO TO 401
      ACHAR = ACHAR+ ((Z(I,1) + Z(I,LL))/2.0) *CX*CY
      BCHAR = BCHAR + ((Z(I,1) + Z(I,LL))/2.0)*SX*CY
  401 ADELT = ADELT + Z(I,J) *CX*CY
      BDELT = BDELT + Z(I,J)*SX*CY
      CDELT = CDELT + Z(I,J) * CX * SY
      DDELT = DDELT + Z(I,J) * SX * SY
   20 CONTINUE
      A(M,N) = (ABLE+ ABAKE+ ACHAR + ADELT) * (4.0/(AKK*ALL))
      B(M,N) = (BCHAR + BDELT) *(4.0/AMK)
      C(M,N) = (CBAKE + CDELT) * (4.0/AMK)
      D(M,N) = (DDELT) * (4.0/AMK)
   22 CONTINUE
C
C     WRITE VALUES OF COEFFICIENTS.
C
      DO 25 M = 1,MMAX
      MM = M-1
      DO 25 N = 1,NMAX
      NN = N-1
      WRITE(6,27)  MM,NN,A(M,N),B(M,N),C(M,N),D(M,N)
   27 FORMAT(1H0,2I5,4F20.4)
   25 CONTINUE
      CALL CLOCK(1)
      CALL EXIT
      END
$ENTRY
```

```
C
C                PROGRAM 2D GENERATOR
C       PROGRAM TO GENERATE DEPENDENT VARIABLE Z AS A FUNCTION
C       OF TWO INDEPENDENT VARIABLES X AND Y.  PROGRAM
C       USES DOUBLE FOURIER EQUATION AND GIVEN COEFFICIENTS.
C       COEFFICIENTS USED IN THIS PARTICULAR PROGRAM ARE
C       TAKEN FROM PRESTON-HARBAUGH PAPER.
C       PROGRAMMED   SEPTEMBER 1965.
C
      DIMENSION A(5,5),B(5,5),C(5,5),D(5,5),T(25,31)
      CALL CLOCK (0)
      DO 100 N=1,5
      DO 100 M=1,5
      FN=N
      FM=M
      A(M,N) = (.3 + .02*FN + .11*FM)*100.
      B(M,N) = (.2 + .04*FN + .08*FM)*100.
      C(M,N) = (.1 + .06*FN + .05*FM)*100.
      D(M,N) = (.08 + .05*FN + .02*FM) *100.
  100 CONTINUE
      DO 101 M =1,5
      C(M,1) = 0.0
      B(1,M) = 0.0
      D(1,M) = 0.0
  101 D(M,1) = 0.0
      PI = 3.14159265
      SUMZ = 0.0
      AH = 15.
      AL = 12.
      WRITE(6,984)
  984 FORMAT(1H1,29HINPUT VALUES OF COEFFICIENTS.)
      WRITE(6,982)
  982 FORMAT(1H0,5X,81HM    N           A(M,N)                 B(M,N)
     1      C(M,N)                  D(M,N))
      DO 300 N = 1,5
      DO 300 M = 1,5
      WRITE(6,981)M,N,A(M,N),B(M,N),C(M,N),D(M,N)
  981 FORMAT(2X,2I5,4F20.8)
  300 CONTINUE
      PIL=PI/AL
      PIH=PI/AH
      XC = -12.
      YC = -15.
      XINC = 1.0
      YINC = 1.0
      WRITE(6,983)
  983 FORMAT(1H1,11X,24HX(I)                  Y(J),
     122H                 Z(I,J),13H         I     J)
      DO 410 I = 1,25
      DO 405 J = 1,31
      DO 200 N=1,5
      X = XC
      Y = YC
      FNT=N-1
      PINH = PIH*FNT
```

```
      DO 200 M=1,5
      FMT=M-1
      PIML = PIL*FMT
      IF(N-1)140,120,140
  120 IF(M-1)150,130,150
  130 AMDA=.25
      GO TO 170
  140 IF(M-1)160,150,160
  150 AMDA=.50
      GO TO 170
  160 AMDA=1.00
  170 ARGX = PIML*X
      ARGY = PINH*Y
      CX = COS(ARGX)
      CY = COS(ARGY)
      SX = SIN(ARGX)
      SY = SIN(ARGY)
      ACC = A(M,N)*CX*CY
      BSC = B(M,N)*SX*CY
      CCS = C(M,N)*CX*SY
      DSS = D(M,N)*SX*SY
      TS = AMDA*(ACC + BSC + CCS + DSS)
      ZC = ZC + TS
  200 CONTINUE
      WRITE(6,980) X,Y,ZC,I,J
  980 FORMAT(1X,3F20.8,2I5)
      WRITE(7,950) X,Y,ZC
  950 FORMAT(3F20.8)
      T(I,J) = ZC
      ZC = 0.0
  405 YC = YC + YINC
      YC = -15.
  410 XC = XC + XINC
      T(1,1) =(T(1,1) + T(1,31) + T(25,1) + T(25,31))/4.0
      T(1,31) = 0.0
      T(25,1) = 0.0
      T(25,31) = 0.0
      DO 700 J = 2,30
      T(1,J) = (T(1,J) + T(25,J))/2.0
      T(25,J) = 0.0
  700 CONTINUE
      DO 701 I = 2,24
      T(I,1) = (T(I,1) + T(I,31))/2.0
      T(I,31) = 0.0
  701 CONTINUE
      DO 702 I = 1,25
      DO 702 J = 1,31
      SUMZ = SUMZ + T(I,J)
  702 CONTINUE
      A00 = (4.0*SUMZ)/720.
      WRITE(6,9999)  SUMZ, A00
 9999 FORMAT(1H1,18HSUM OF Z VALUES = ,F20.8,5X,9HA(0,0) = ,F20.8)
      CALL CLOCK (1)
  490 CALL EXIT
      END
$ENTRY
```


START
WRITE: TITLE and HEADINGS
READ: EL, H, KK, LL
READ: X(I), Y(J), Z(I,J)
NMAX = NO. HARMONICS
MMAX = NO. HARMONICS
PIE = 3.14159265
AKK = KK-1
ALL = LL-1
MKK = AKK
MLL = ALL
AMK = AKK * ALL
22 M = 1,MMAX
FM = M-1
22 N = 1,NMAX
FN = N-1
ABLE = 0.
ADELT = 0.
ABAKE = 0.
ACHAR = 0.
BCHAR = 0.
BDELT = 0.
CBAKE = 0.
CDELT = 0.
DDELT = 0.
A
B
B


A
PIM = PIE * FM
PIN = PIE * FN
ABLE = ((((Z(1,1) + Z(KK,1))/2.0)
((Z(1,LL) + Z(KK,LL))/2.0))/
2.00 * COS (PIM) * COS (PIN)
20
I = 2,MKK
20
J = 2,MLL
ARGX = PIM * X(I)/EL
ARGY = PIN * X(I)/H
CX = COS(ARGX)
CY = COS(ARGY)
SX = SIN(ARGX)
SY = SIN(ARGY)
B
B
I:2
>
≤
ABAKE ABAKE ((Z(I,J)
Z(KK,J))/2.0) * CX * CY
CBAKE CBAKE ((Z(I,J)
Z(KK,J))/2.0) * CX * SY
J:2
>
≤
ACHAR ACHAR ((Z(I,1)
Z(I,LL))/2.0) * CX * CY
BCHAR BCHAR ((Z(I,1)
Z(I,LL))/2.0) * SX * CY
B


B
B
B
ADELT = ADELT + Z(I,J) * CX * CY
BDELT = BDELT + Z(I,J) * SX * CY
CDELT = CDELT + Z(I,J) * CX * SY
DDELT = DDELT + Z(I,J) * SX * SY
20
A(M,N) = (ABLE + ABAKE + ACHAR + ADELT) * (4./AKK * ALL)
B(M,N) = (BCHAR + BDELT) * (4./AMK)
C(M,N) = (CBAKE + CDELT) * (4./AMK)
D(M,N) = (DDELT) * (4./AMK)
A
A
22
25
M = 1,MMAX
MM = M-1
25
N = 1,NMAX
NN = N-1
WRITE: MM, NN, A(M,N), B(M,N), C(M,N), D(M,N)
25
CALL EXIT
END

COMPUTER PROGRAM TO DETERMINE COVARIANCE FUNCTION AND POWER SPECTRUM

Purpose

The purpose of this program is to determine the covariance function and the power spectrum of a one-dimensional function, f(x). The secondary purpose is to plot the power spectrum and the confidence bands of the spectrum, to facilitate analysis of the function.

Language

Fortran IV (IBM 7040 Computer)

Symbolic Dictionary

Variable	S/A*	I/O**	Description
X	A	I	Value of the function at each data point. Read in as fixed point, mx.
I	S	--	Series index.
K	S	--	Lag Index.
C	A	O	Covariance estimate.
L	A	O	Raw estimate of the power spectrum.
U	A	O	Smoothed estimate of the power spectrum.

---

* S--Single Variable; A--Array of Variables

** I--Input Variable: O--Output Variables

Variable	S/A	I/O	Description
N	S	I	Number of data points.
M	S	I	Maximum lag. (Less than N/3)
EDF	A	I	Equivalent degrees of freedom.
TL95	A	I	Table of confidence limits, lower 95 per cent.
TL90	A	I	Table of confidence limits, lower 90 per cent.
TU95	A	I	Table of confidence limits, upper 95 per cent.
TU90	A	I	Table of confidence limits, upper 90 per cent.
KOW	S	I	Number of samples.
KOUNT	S	--	Sample index.
LA	A	I	Storage for data identification.
LAG	S	O	Degree of offset by which the function is correlated with itself, to determine covariance estimate.
UMAX	S	--	Maximum value of smoothed power spectrum in each frequency band.
UMIN	S	--	Minimum value of power spectrum.
SPANA	S	--	90 per cent confidence band of smoothed power spectrum.
SPANB	S	--	95 per cent confidence band.

Program Routine

This program utilizes data points which represent the value of the function to be analyzed at equally spaced intervals along the function. Using a numerical integration

technique, the trapezoidal rule, the values of the covariance estimate are determined according to Equation (3.25) of this paper. A rough estimate of the power spectrum is computed using Equation(3.26) and smoothed according to Equation (3.27). The table of confidence limits presented by Granger[77] is read into the program and the ninety and ninety-five per cent confidence bands are determined. The smoothed power spectrum with these confidence bands is displayed graphically.

```
C       -------------------------------------------------------------
C
C       PROGRAM  COVAR   FOR COMPUTING COVARIANCE FUNCTION
C       AND POWER SPECTRUM.  PROGRAMMED SEPT. 1965 BY
C       FW PRESTON AND WD ALDENDERFER.
C
C       -------------------------------------------------------------
C     NOMENCLATURE
C     K      = LAG INDEX
C     I      = SERIES INDEX
C     J      = WEIGHTING FACTOR INDEX
C     C(K)   = COVARIANCE ESTIMATE
C     L(J)   = RAW ESTIMATE OF POWER SPECTRUM
C     U(J)   = SMOOTHED ESTIMATE OF POWER SPECTRUM
C     M      = MAXIMUM LAG (LESS THAN N/3)
C     N      = NUMBER OF DATA POINTS
C     LA(I)  = STORAGE FOR DATA IDENTIFICATION
C     EDF(I) = EQUIVALENT DEGREES OF FREEDOM
C     TL95(I)= TABLE OF CONFIDENC LIMITS - LOWER- 95PERCENT
C     TL90(I)=                           - LOWER- 90PERCENT
C     TU90(I)=                           - UPPER 90 PERCENT
C     TU95(I)=                           - UPPER 95 PERCENT
C     KOW IS THE NUMBER OF SAMPLES
C
C
      DIMENSION  X(3200),C(301),L(302),U(301),EDF(13),TL95(13),
     1TU90(13),TU95(13),LA(13),JOW1(53),JOW2(53),TL90(13),MX(3200)
      REAL L
      CALL CLOCK  (0)
      KOW = 2
      KOUNT = 1
      READ (5,110) (EDF(I),TL95(I),TL90(I),TU90(I),TU95(I),I=1,13)
  110 FORMAT(F5.1,4F5.2)
  400 READ (5,99)(LA(I),I=1,13)
   99 FORMAT(13A6)
      READ (5,100) N,M
  100 FORMAT (2I5)
      READ (5,101) (MX(I),I=1,N)
  101 FORMAT(80I1)
      DO 700 II= 1,N
      X(II) = MX(II)
  700 CONTINUE
      DO 10 I = 1,N
      IF (X(I) - 0. )11,11,10
   11 X(I) = -1.0
   10 CONTINUE
C
C     COMPUTING COVARIANCE ESTIMATE  C(K)
C
      FM = M
      MP1 = M+1
      FN =N
      J = 0
      AK = 2.0*FN/FM
      DO 910 I = 1,13
      J = J+1
      IF(AK.LT.EDF(I))  GO TO 320
  910 CONTINUE
```

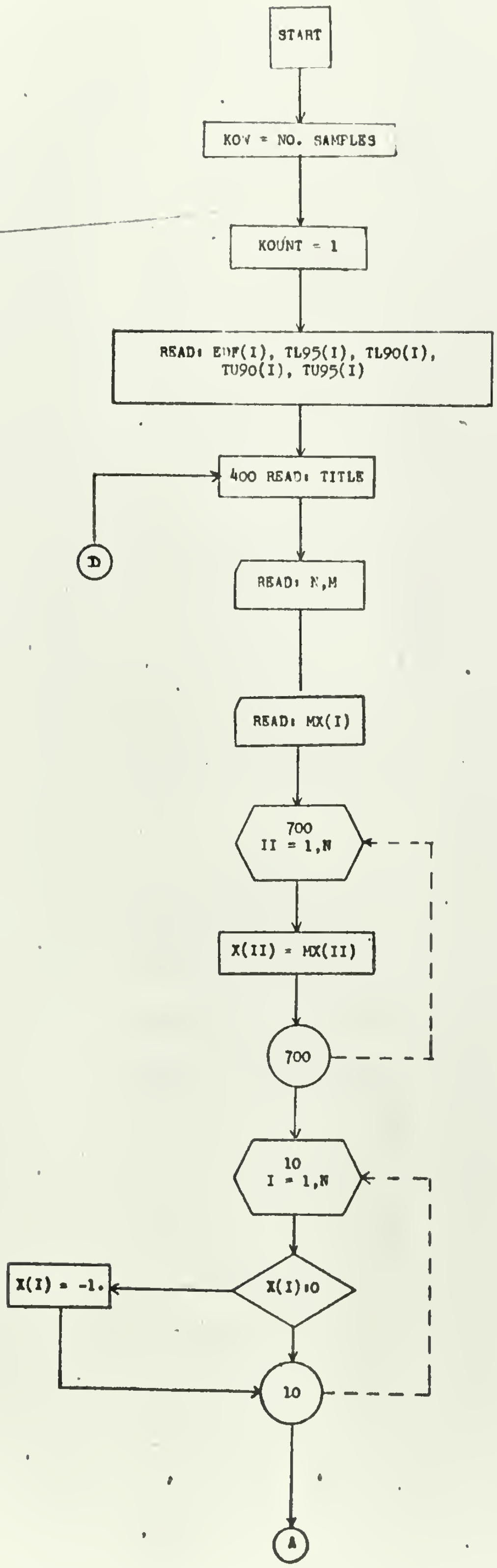
```
  320 J = J-1
      CHG = (AK-EDF(J))/(EDF(J+1)-EDF(J))
      T90L = TL90(J) + (TL90(J+1)-TL90(J))*CHG
      T95L = TL95(J) + (TL95(J+1)-TL95(J))*CHG
      T95U = TU95(J) + (TU95(J+1)-TU95(J))*CHG
      T90U = TU90(J) + (TU90(J+1)-TU90(J))*CHG
      DO 210 KK = 1,MP1
      K = KK-1
      S1= 0.0
      NMK =N-K
      FNMK = NMK
      KP1  = K+1
      DO 205 I=1,NMK
      IPK =I+K
  205 S1 = S1 + X(I)*X(IPK)
      S2 = 0.0
      DO 206 I=KP1,N
  206 S2 = S2 + X(I)
      S3 = 0.0
      DO 207 I=1,NMK
  207 S3 = S3 + X(I)
  210 C(KK) = (S1 - (S2*S3)/FNMK)/FNMK
C
C     COMPUTATION OF RAW ESTIMATE OF POWER SPECTRUM L(J)
C
      MN1= M-1
      DO 230 JJ = 1,MP1
      J= JJ-1
      FJ=J
      S1 =0.0
      DO 220 K=1,MN1
      FK=K
  220 S1 = S1 + C(K+1)*COS(3.1415926*FK*FJ/FM)
  230 L(JJ+1) =(2.0*S1 + C(1) + C(M+1)*COS(3.1415926*FJ))/6.2831952
      L(1)=L(3)
      L(M+3) = L(M+1)
C
C     COMPUTATION OF SMOOTHED POWER SPECTRUM ESTIMATE
C
      MP2 = M+ 2
      DO 240 J =2,MP2
  240 U(J) = .25*L(J-1) + .50*L(J) + .25*L(J+1)
C
C     WRITE HEADINGS
C
      WRITE(6,103)
  103 FORMAT(1H1,36X,25HPOWER SPECTRUM ESTIMATION/
     137X,24HFOR DISCRETE TIME SERIES/31X,
     237HUSING TUKEY HANNING WEIGHTING FACTORS///)
      WRITE(6,104)(LA(I),I=1,13)
  104 FORMAT(1X,13A6///)
      WRITE(6,106)
C
C     WRITE POWER SPECTRUM AND TOLERANCE ESTIMATES
C
```

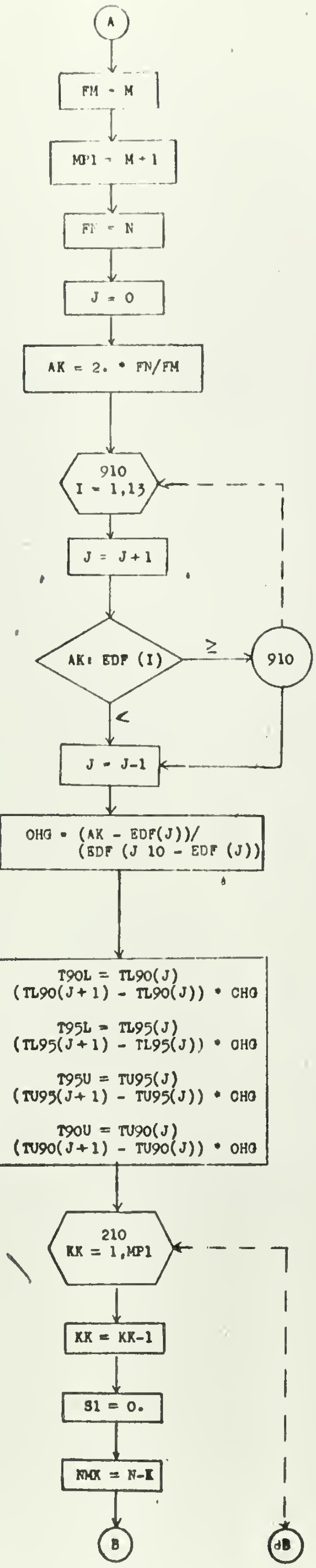
```
      DO 250 I =2,MP2
      LAG = I- 2
      U90L = T90L*U(I)
      U95L = T95L*U(I)
      U90U = T90U*U(I)
      U95U = T95U*U(I)
  250 WRITE(6,105) LAG,C( I1),L(I),U90L,U(I),U95U,U95L,U(I),U90U
  105 FORMAT(1X,I3,8(2X,F10.4))
C
C                PLOTTING ROUTINE
C
  764 ISTAR   =-13702990896
      IMINUS = 29809118256
      IPLUS  = 17997958192
      IBLANK = -17997958192
      II     = 27661634608
      WRITE (6,103)
      WRITE (6,104)(LA(I),I=1,13)
      WRITE (6,107)
  107 FORMAT(19X,23HSMOOTHED POWER SPECTRUM,34X,
     123HSMOOTHED POWER SPECTRUM /15X,
     230HWITH 90 PCT. CONFIDENCE LIMITS,27X,
     330HWITH 95 PCT. CONFIDENCE LIMITS/6X,
     451H0    1    2    3    4.   5    6    7    8    9    0,
     56X,51H0    1    2    3    4    5    6    7    8    9    0)
  106 FORMAT(1H0,39X,
     150HSMOOTHED SPECTRUM WITH INDICATED CONFIDENCE LIMITS/
     234X,7H90 PCT.,17X,7H90 PCT.,5X,7H95 PCT.,17X,7H95 PCT./8X,
     310HCOVARIANCE,5X,3HRAW,9X,5HLOWER,19X,5HUPPER,7X,
     45HLOWER,19X,5HUPPER/1X,3HLAG,5X,8HSPECTRUM,4X,8HSPECTRUM,
     56X,5HLIMIT,6X,7HAVERAGE,6X,5HLIMIT,7X,5HLIMIT,6X,7HAVERAGE,
     66X,5HLIMIT)
C
C     PLOTTING OF SPECTRUM
C
      DO 280 I=1,53
      JOW1(I)=IMINUS
  280 JOW2(I)=IMINUS
      WRITE(6,108) (JOW1(I),I=1,53),(JOW2(I),I=1,53)
  108 FORMAT(5X,53A1,5X,53A1)
      DO 290 I=2,52
      JOW1(I)=IBLANK
  290 JOW2(I)=IBLANK
      JOW1(1)=II
      JOW1(53)=II
      JOW2(1) =II
      JOW2(53)=II
      UMAX = U(2)
      UMIN = U(2)
      DO 300 I = 3,MP2
      IF(U(I).GT.UMAX) UMAX=U(I)
      IF(U(I).LT.UMIN) UMIN=U(I)
  300 CONTINUE
      SPANA = ALOG10(UMAX*T90U) - ALOG10(UMIN*T90L)
      SPANB = ALOG10(UMAX*T95U) - ALOG10(UMIN*T95L)
```

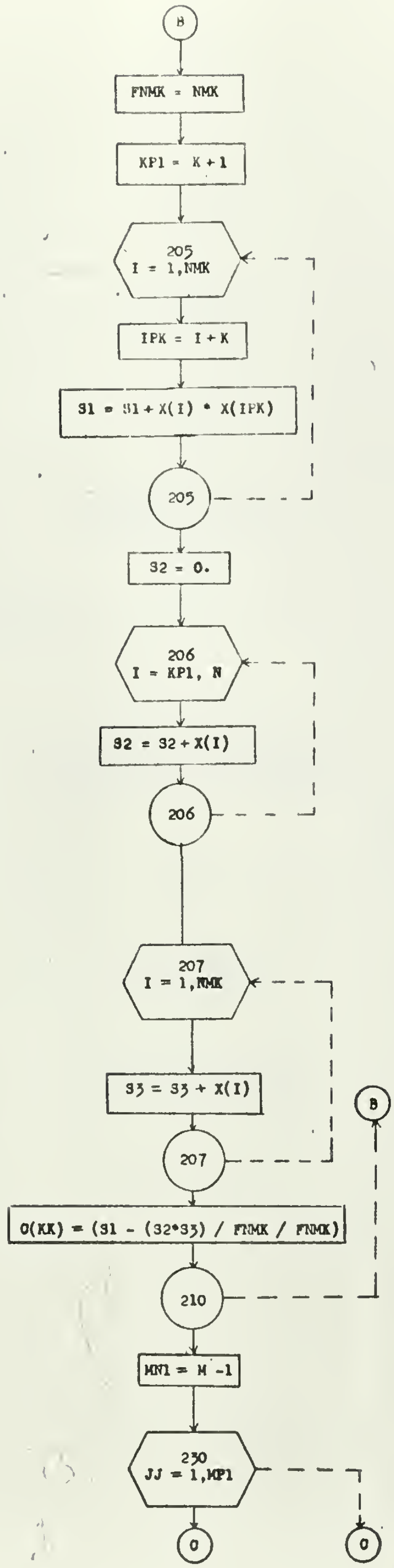
```
      ZA = ALOG10(UMIN*T90L)
      ZB = ALOG10(UMIN*T95L)
      DO 310 I =2,MP2
      IA1=((ALOG10(U(I)*T90L) - ZA)/SPANA)*50. + 2.0
      IA2=((ALOG10(U(I))-ZA)/SPANA)*50. + 2.0
      IA3=((ALOG10(U(I)*T90U)- ZA)/SPANA)*50. + 2.0
      JOW1(IA1) = IMINUS
      JOW1(IA2) =ISTAR
      JOW1(IA3) =IPLUS
      IB1=((ALOG10(U(I)*T95L) - ZB)/SPANB)*50.+ 2.0
      IB2 = ((ALOG10(U(I))-ZB)/SPANA)*50. + 2.0
      IB3=((ALOG10(U(I)*T95U)-ZB)/SPANB)*50. + 2.0
      JOW2(IB1) = IMINUS
      JOW2(IB2) = ISTAR
      JOW2(IB3) =IPLUS
      LAG = I- 2
      WRITE(6,109) LAG,(JOW1(J),J=1,53),LAG,(JOW2(J),J=1,53)
  109 FORMAT(1X,I3,1X,53A1,1X,I3,1X,53A1)
      JOW1(IA1) = IBLANK
      JOW1(IA2) = IBLANK
      JOW1(IA3) = IBLANK
      JOW2(IB1) = IBLANK
      JOW2(IB2) = IBLANK
      JOW2(IB3) = IBLANK
  310 CONTINUE
C
C            END OF PLOTTING ROUTINE
C
  167 IF(KOUNT.EQ.KOW)  GO TO 765
      CALL CLOCK  (2)
      KOUNT = KOUNT + 1
      GO TO 400
  765 CALL EXIT
      END
$ENTRY
  2.    .05   .10 2.30 2.99
  3.    .12   .20 2.08 2.60
  4.    .18   .26 1.94 2.37
  5.    .23   .32 1.85 2.21
  6.    .27   .37 1.77 2.10
  8.    .34   .44 1.68 1.94
 10.    .39   .49 1.60 1.83
 12.    .43   .53 1.55 1.75
 15.    .48   .57 1.48 1.66
 20.    .54   .62 1.42 1.51
 30.    .62   .69 1.34 1.46
 50.    .69   .75 1.26 1.34
 100.   .77   .82 1.18 1.22
```

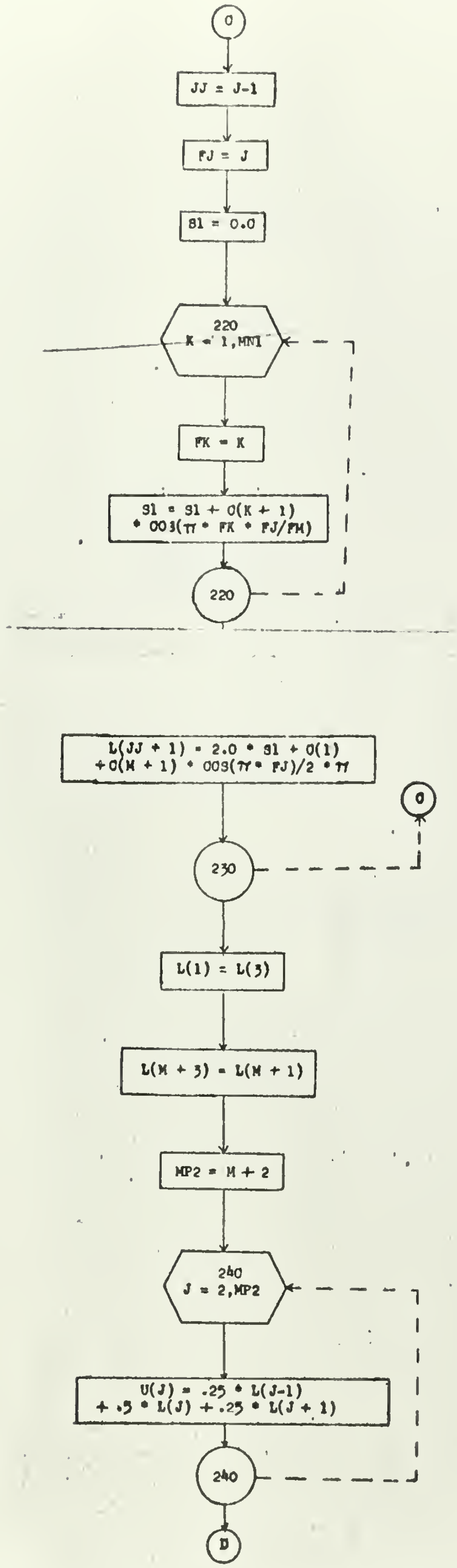
PROGRAM TO DETERMINE COVARIANCE

FUNCTION AND POWER SPECTRUM

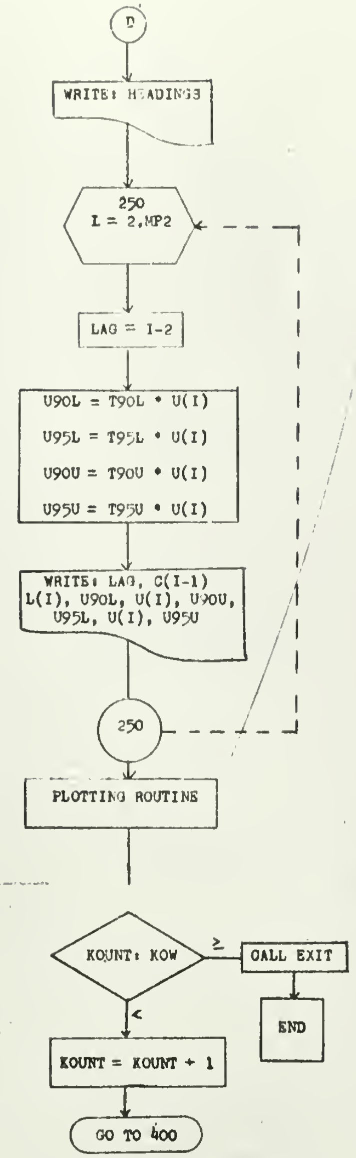



A
FM = M
MP1 = M + 1
FN = N
J = 0
AK = 2. * FN/FM
910
I = 1,13
J = J + 1
AK : EDF (I)
≥
910
<
J = J-1
OHG = (AK - EDF(J))/
(EDF (J 10 - EDF (J))
T90L = TL90(J)
(TL90(J+1) - TL90(J)) * OHG
T95L = TL95(J)
(TL95(J+1) - TL95(J)) * OHG
T95U = TU95(J)
(TU95(J+1) - TU95(J)) * OHG
T90U = TU90(J)
(TU90(J+1) - TU90(J)) * OHG
210
KK = 1,MP1
KK = KK-1
S1 = 0.
NMK = N-K
B
dB

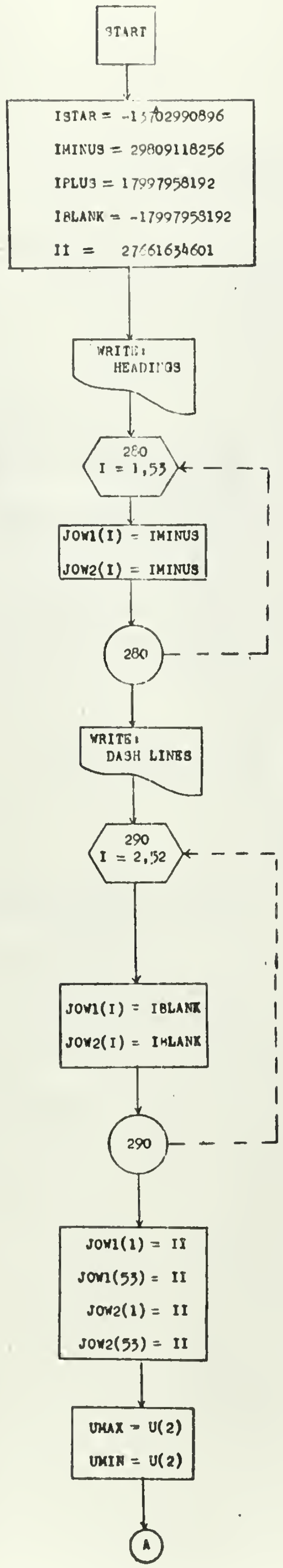

B
FNMK = NMK
KP1 = K + 1
205
I = 1,NMK
IPK = I + K
S1 = S1 + X(I) * X(IPK)
205
S2 = 0.
206
I = KP1, N
S2 = S2 + X(I)
206
207
I = 1,NMK
S3 = S3 + X(I)
B
207
C(KK) = (S1 - (S2*S3) / FNMK / FNMK)
210
MN1 = M -1
230
JJ = 1,KP1
C
C

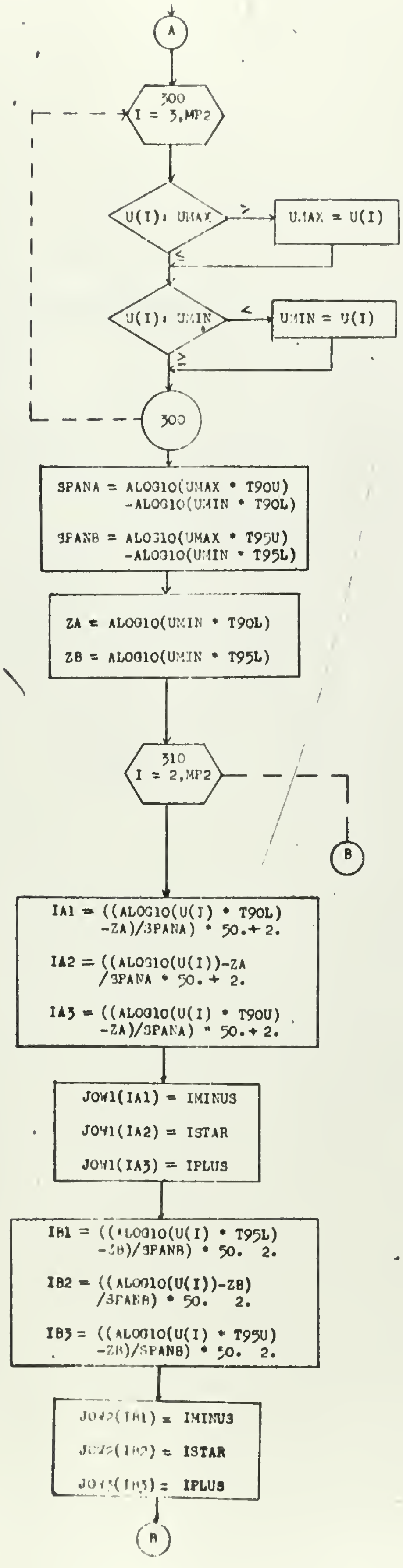


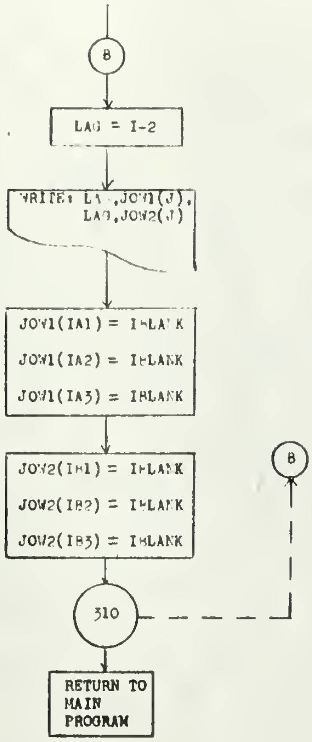
O
JJ = J-1
FJ = J
S1 = 0.0
220
K = 1,MN1
FK = K
S1 = S1 + C(K + 1)
* COS(π * FK * FJ/FM)
220
L(JJ + 1) = 2.0 * S1 + C(1)
+ C(M + 1) * COS(π * FJ)/2 * π
O
230
L(1) = L(3)
L(M + 3) = L(M + 1)
MP2 = M + 2
240
J = 2,MP2
U(J) = .25 * L(J-1)
+ .5 * L(J) + .25 * L(J + 1)
240
D


D
WRITE: HEADINGS
250
I = 2,MP2
LAG = I-2
U90L = T90L * U(I)
U95L = T95L * U(I)
U90U = T90U * U(I)
U95U = T95U * U(I)
WRITE: LAG, C(I-1)
L(I), U90L, U(I), U90U,
U95L, U(I), U95U
250
PLOTTING ROUTINE
KOUNT: KOW
≥
CALL EXIT
END
<
KOUNT = KOUNT + 1
GO TO 400

"COVAR" PLOTTING ROUTINE

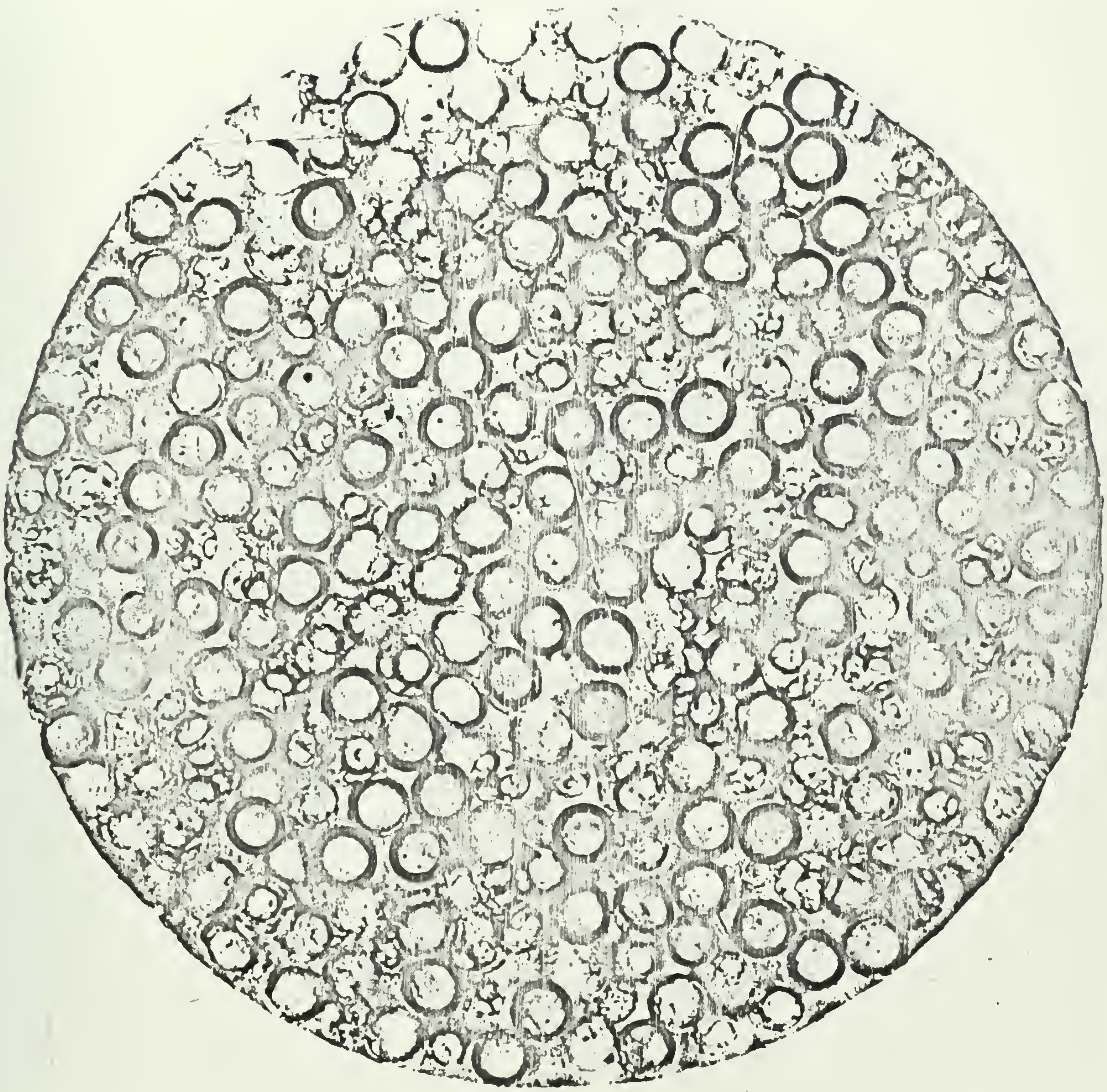



A
300
I = 3,MP2
U(I): UMAX
>
UMAX = U(I)
≤
U(I): UMIN
<
UMIN = U(I)
≥
300
SPANA = ALOG10(UMAX * T90U)
-ALOG10(UMIN * T90L)
SPANB = ALOG10(UMAX * T95U)
-ALOG10(UMIN * T95L)
ZA = ALOG10(UMIN * T90L)
ZB = ALOG10(UMIN * T95L)
310
I = 2,MP2
B
IA1 = ((ALOG10(U(I) * T90L)
-ZA)/SPANA) * 50.+ 2.
IA2 = ((ALOG10(U(I))-ZA
/SPANA * 50. + 2.
IA3 = ((ALOG10(U(I) * T90U)
-ZA)/SPANA) * 50.+ 2.
JOW1(IA1) = IMINUS
JOW1(IA2) = ISTAR
JOW1(IA3) = IPLUS
IB1 = ((ALOG10(U(I) * T95L)
-ZB)/SPANB) * 50. 2.
IB2 = ((ALOG10(U(I))-ZB)
/SPANB) * 50. 2.
IB3 = ((ALOG10(U(I) * T95U)
-ZB)/SPANB) * 50. 2.
JOW2(IB1) = IMINUS
JOW2(IB2) = ISTAR
JOW2(IB3) = IPLUS
B

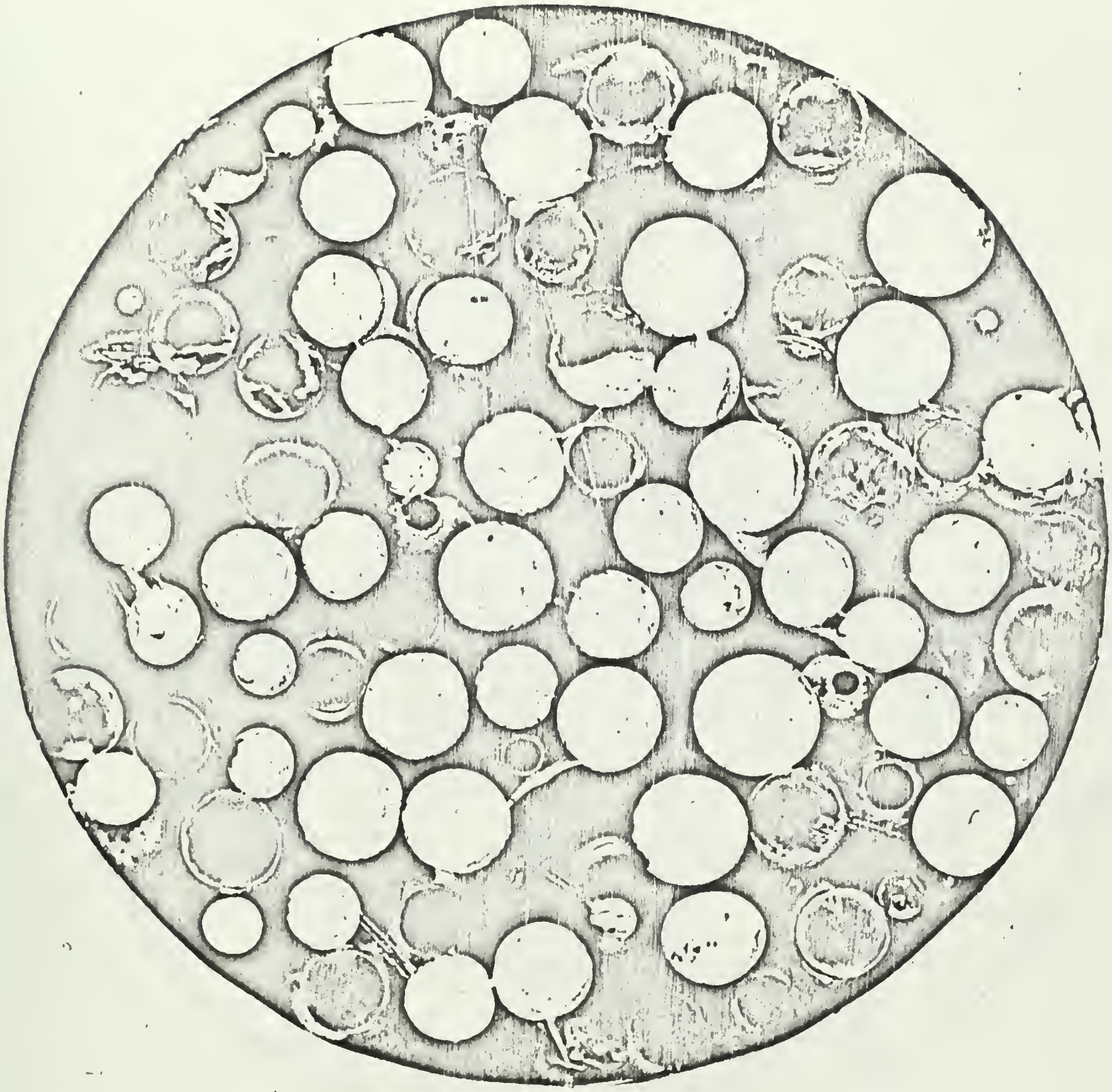

B
LAG = I-2
WRITE: LAG,JOW1(J),
LAG,JOW2(J)
JOW1(IA1) = IBLANK
JOW1(IA2) = IBLANK
JOW1(IA3) = IBLANK
JOW2(IB1) = IBLANK
JOW2(IB2) = IBLANK
JOW2(IB3) = IBLANK
310
B
RETURN TO
MAIN
PROGRAM

APPENDIX D

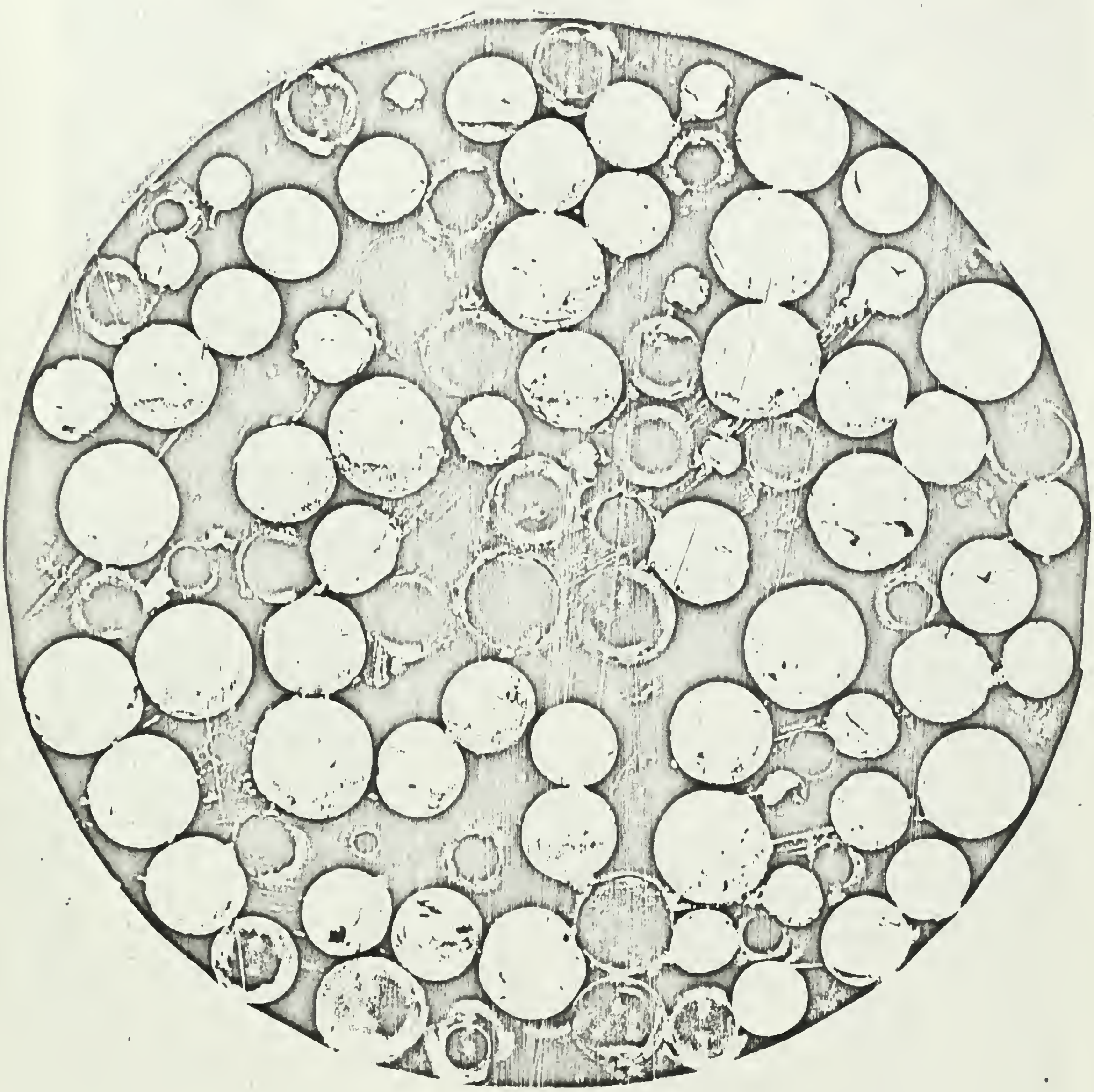
PICTURES



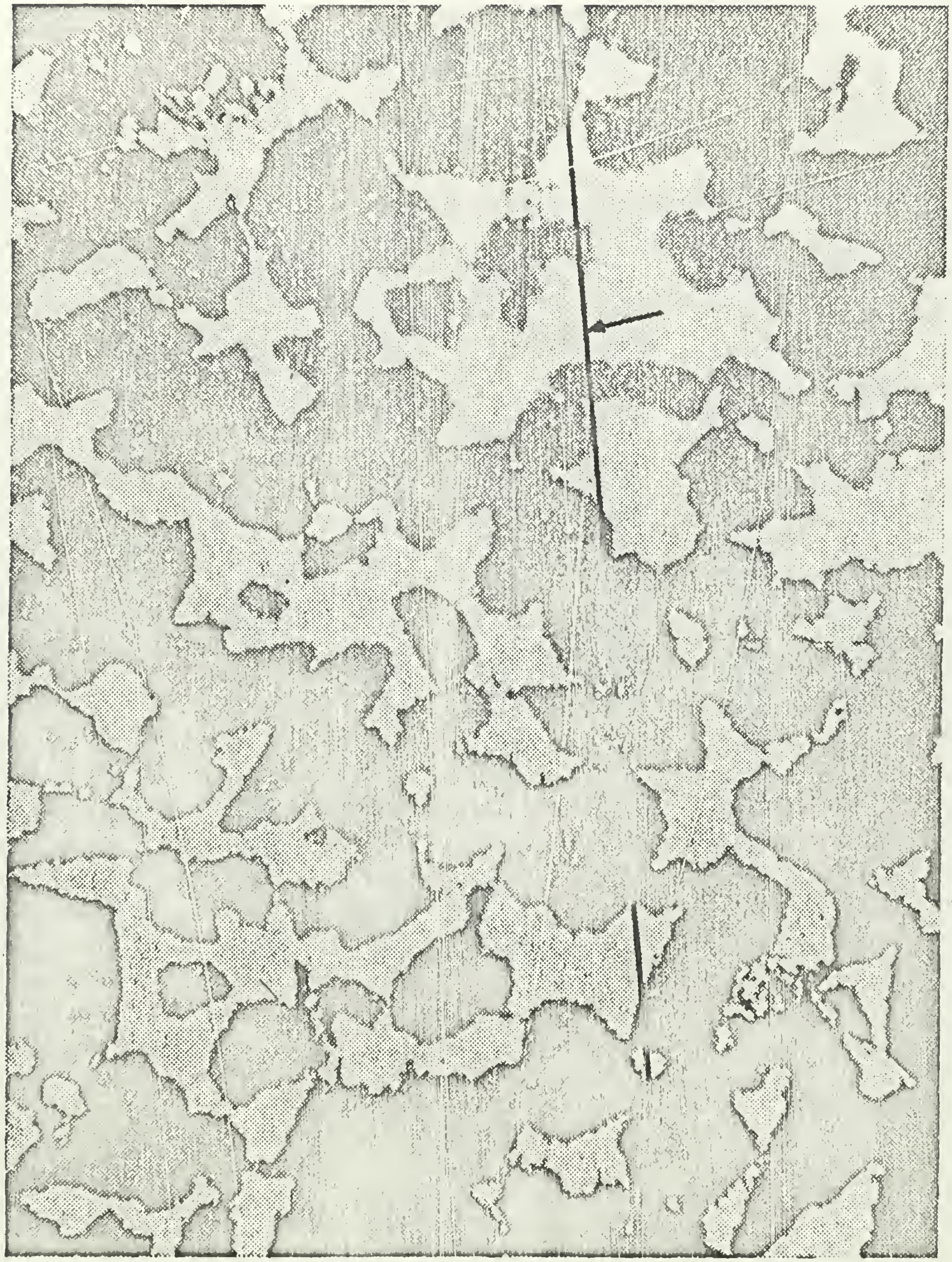
Sample X Three Millimeter Beads



Sample Y Two Size (4mm. and 5mm.) Beads



Sample Z Three Size (3mm., 4mm., and 5mm.) Beads



Photomicrograph of porous medium (sandstone) 92X. Courtesy of Humble Oil and Refining Company. Porosity 26.3 per cent. The line is 0.17 cm in length.

APPENDIX E RESULTS OF AUTOCOVARIANCE PROGRAM

POWER SPECTRUM ESTIMATION
FOR DISCRETE TIME SERIES
USING TUKEY-HANNING WEIGHTING FACTORS

TWO SIZE BEADS 40 ROWS MIXED LAG = 40

(1)

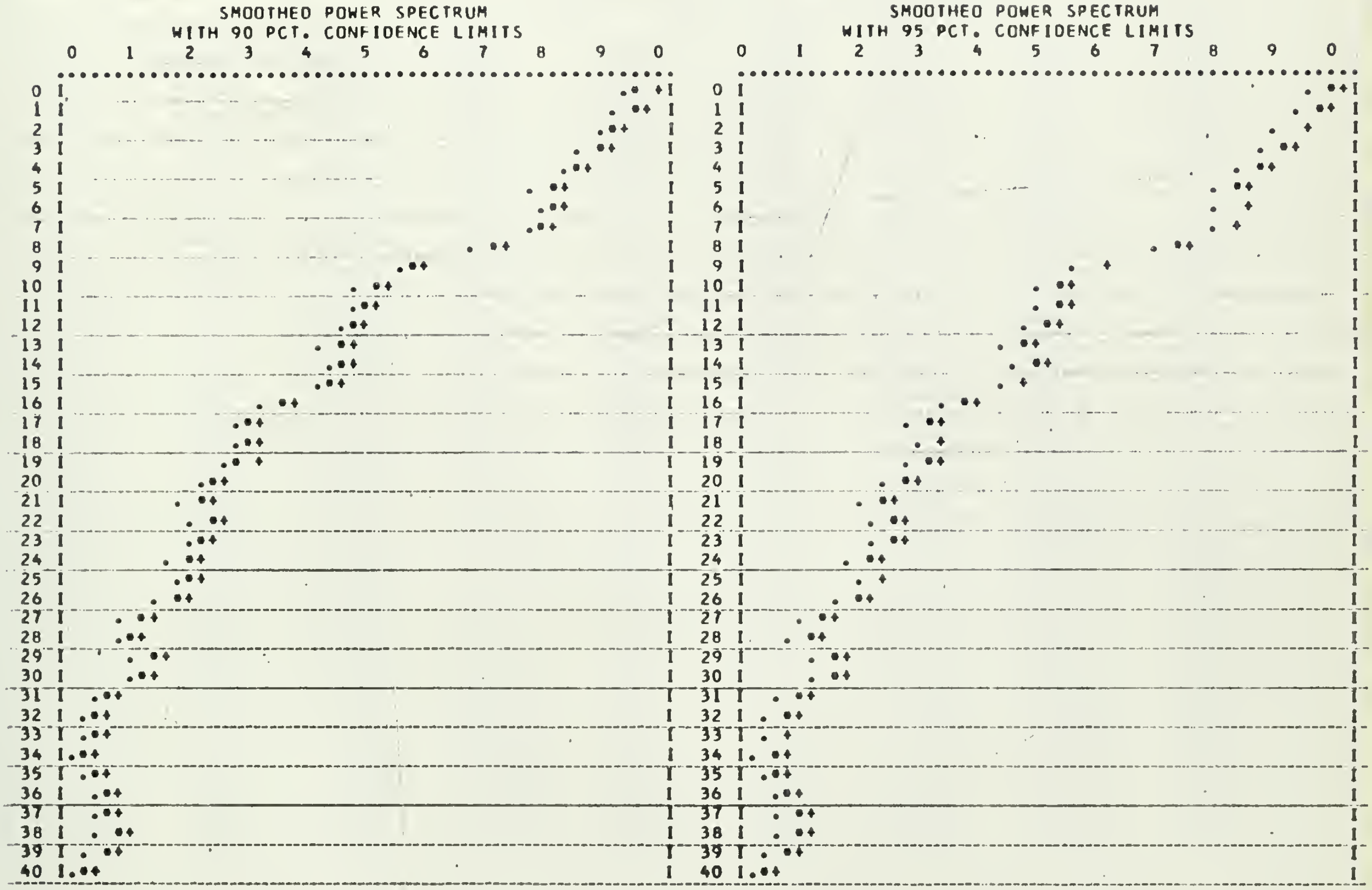
SMOOTHED SPECTRUM WITH INDICATED CONFIDENCE LIMITS

LAG	COVARIANCE SPECTRUM	RAW SPECTRUM	90 PCT. LOWER LIMIT	AVERAGE	90 PCT. UPPER LIMIT	95 PCT. LOWER LIMIT	AVERAGE	95 PCT. UPPER LIMIT
0	0.9409	0.7021	0.6647	0.7353	0.7911	0.6367	0.7353	0.7970
1	0.6656	0.7684	0.6291	0.6959	0.7488	0.6027	0.6959	0.7544
2	0.4372	0.5449	0.5591	0.6185	0.6655	0.5356	0.6185	0.6705
3	0.2795	0.6159	0.5069	0.5607	0.6033	0.4856	0.5607	0.6078
4	0.1668	0.4663	0.4365	0.4829	0.5196	0.4182	0.4829	0.5234
5	0.0853	0.3831	0.3751	0.4149	0.4464	0.3593	0.4149	0.4497
6	0.0401	0.4270	0.3815	0.4220	0.4541	0.3654	0.4220	0.4574
7	0.0053	0.4507	0.3580	0.3960	0.4261	0.3430	0.3960	0.4293
8	-0.0057	0.2556	0.2522	0.2790	0.3002	0.2416	0.2790	0.3024
9	0.0084	0.1539	0.1543	0.1707	0.1837	0.1478	0.1707	0.1851
10	0.0375	0.1195	0.1187	0.1313	0.1413	0.1137	0.1313	0.1424
11	0.0604	0.1325	0.1165	0.1289	0.1387	0.1116	0.1289	0.1397
12	0.0577	0.1310	0.1081	0.1195	0.1286	0.1035	0.1195	0.1296
13	0.0475	0.0836	0.0951	0.1052	0.1132	0.0911	0.1052	0.1140
14	0.0160	0.1227	0.0999	0.1105	0.1189	0.0957	0.1105	0.1198
15	-0.0068	0.1130	0.0919	0.1017	0.1094	0.0880	0.1017	0.1102
16	-0.0170	0.0580	0.0652	0.0721	0.0776	0.0624	0.0721	0.0782
17	-0.0260	0.0594	0.0531	0.0588	0.0633	0.0509	0.0588	0.0637
18	-0.0325	0.0583	0.0538	0.0595	0.0640	0.0515	0.0595	0.0645
19	-0.0214	0.0620	0.0515	0.0569	0.0613	0.0493	0.0569	0.0617
20	-0.0027	0.0455	0.0431	0.0477	0.0514	0.0413	0.0477	0.0517
21	0.0354	0.0380	0.0388	0.0430	0.0462	0.0372	0.0430	0.0466
22	0.0533	0.0504	0.0418	0.0463	0.0498	0.0401	0.0463	0.0502
23	0.0365	0.0464	0.0400	0.0442	0.0476	0.0383	0.0442	0.0479
24	0.0242	0.0337	0.0361	0.0399	0.0430	0.0346	0.0399	0.0433
25	0.0232	0.0458	0.0366	0.0405	0.0436	0.0351	0.0405	0.0439
26	0.0122	0.0365	0.0335	0.0370	0.0398	0.0321	0.0370	0.0401
27	0.0137	0.0292	0.0268	0.0296	0.0318	0.0256	0.0296	0.0321
28	0.0032	0.0235	0.0250	0.0276	0.0297	0.0239	0.0276	0.0299
29	-0.0079	0.0343	0.0284	0.0314	0.0338	0.0272	0.0314	0.0340
30	-0.0259	0.0335	0.0275	0.0304	0.0327	0.0263	0.0304	0.0329
31	-0.0262	0.0204	0.0223	0.0247	0.0266	0.0214	0.0247	0.0268
32	-0.0133	0.0246	0.0207	0.0229	0.0246	0.0198	0.0229	0.0248
33	-0.0067	0.0221	0.0201	0.0222	0.0239	0.0192	0.0222	0.0241
34	-0.0127	0.0201	0.0189	0.0209	0.0224	0.0181	0.0209	0.0226
35	-0.0162	0.0212	0.0200	0.0221	0.0237	0.0191	0.0221	0.0239
36	-0.0273	0.0258	0.0215	0.0238	0.0256	0.0206	0.0238	0.0257
37	-0.0333	0.0222	0.0221	0.0244	0.0263	0.0211	0.0244	0.0265
38	-0.0330	0.0274	0.0229	0.0254	0.0273	0.0220	0.0254	0.0275
39	-0.0391	0.0245	0.0209	0.0232	0.0249	0.0201	0.0232	0.0251
40	-0.0413	0.0163	0.0184	0.0204	0.0219	0.0176	0.0204	0.0221

POWER SPECTRUM ESTIMATION
FOR DISCRETE TIME SERIES
USING TUKEY-HANNING WEIGHTING FACTORS

TWO SIZE BEADS 40 ROWS MIXED LAG = 40

(1)



2 TIME 1.3569 MIN

POWER SPECTRUM ESTIMATION
FOR DISCRETE TIME SERIES
USING TUKEY-HANNING WEIGHTING FACTORS

TWO SIZE BEADS 40 ROWS MIXED LAG = 40

(2)

SMOOTHED SPECTRUM WITH INDICATED CONFIDENCE-LIMITS

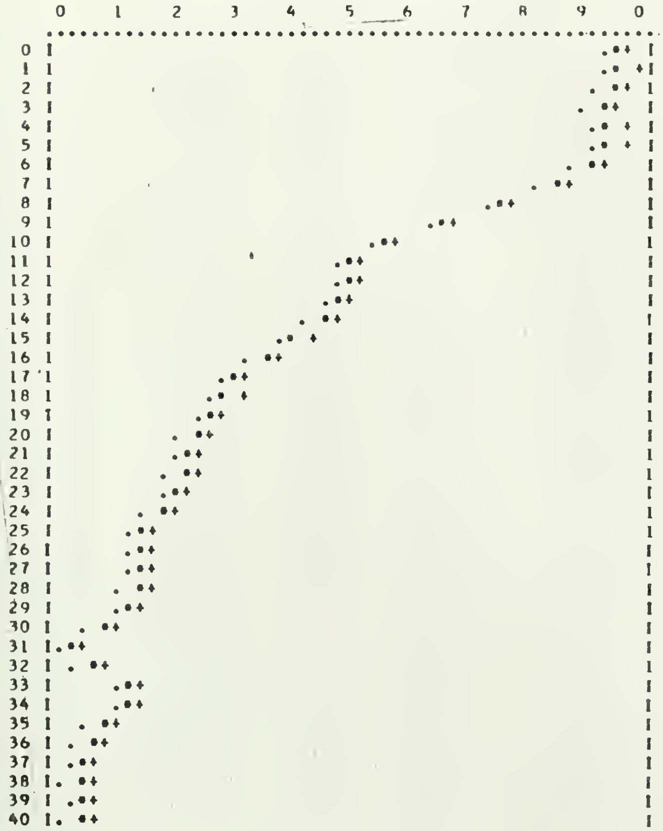
LAG	COVARIANCE SPECTRUM	RAW SPECTRUM	90 PCT. LOWER LIMIT	AVERAGE	90 PCT. UPPER LIMIT	95 PCT. LOWER LIMIT	AVERAGE	95 PCT. UPPER LIMIT
0	0.9748	0.6457	0.5549	0.6138	0.6605	0.5316	0.6138	0.6654
1	0.7058	0.5820	0.5712	0.6318	0.6798	0.5472	0.6318	0.6849
2	0.4705	0.7177	0.5446	0.6025	0.6483	0.5217	0.6025	0.6531
3	0.2788	0.3926	0.4995	0.5526	0.5946	0.4785	0.5526	0.5990
4	0.1308	0.7075	0.5338	0.5905	0.6354	0.5114	0.5905	0.6401
5	0.0277	0.5545	0.5319	0.5884	0.6331	0.5095	0.5884	0.6378
6	-0.0383	0.5369	0.4693	0.5191	0.5585	0.4495	0.5191	0.5627
7	-0.0681	0.4480	0.3783	0.4184	0.4502	0.3624	0.4184	0.4536
8	-0.0766	0.2408	0.2665	0.2948	0.3172	0.2553	0.2948	0.3195
9	-0.0500	0.2495	0.1885	0.2085	0.2243	0.1805	0.2085	0.2260
10	-0.0153	0.0941	0.1307	0.1446	0.1556	0.1252	0.1446	0.1567
11	0.0219	0.1406	0.1060	0.1172	0.1262	0.1015	0.1172	0.1271
12	0.0435	0.0937	0.1035	0.1145	0.1232	0.0991	0.1145	0.1241
13	0.0376	0.1299	0.0986	0.1090	0.1173	0.0944	0.1090	0.1182
14	0.0379	0.0826	0.0871	0.0963	0.1036	0.0834	0.0963	0.1044
15	0.0214	0.0902	0.0745	0.0824	0.0887	0.0714	0.0824	0.0894
16	0.0149	0.0668	0.0619	0.0685	0.0737	0.0593	0.0685	0.0742
17	0.0164	0.0501	0.0501	0.0555	0.0597	0.0480	0.0555	0.0601
18	0.0054	0.0548	0.0483	0.0534	0.0574	0.0462	0.0534	0.0579
19	0.0020	0.0537	0.0447	0.0495	0.0532	0.0428	0.0495	0.0536
20	-0.0065	0.0356	0.0395	0.0437	0.0470	0.0378	0.0437	0.0474
21	-0.0105	0.0500	0.0384	0.0424	0.0457	0.0368	0.0424	0.0460
22	-0.0158	0.0343	0.0369	0.0408	0.0439	0.0354	0.0408	0.0443
23	-0.0268	0.0448	0.0355	0.0393	0.0423	0.0340	0.0393	0.0426
24	-0.0227	0.0333	0.0322	0.0356	0.0383	0.0308	0.0356	0.0386
25	-0.0250	0.0309	0.0280	0.0310	0.0333	0.0268	0.0310	0.0336
26	-0.0252	0.0289	0.0281	0.0311	0.0334	0.0269	0.0311	0.0337
27	-0.0305	0.0358	0.0283	0.0313	0.0337	0.0271	0.0313	0.0340
28	-0.0466	0.0250	0.0275	0.0304	0.0327	0.0263	0.0304	0.0329
29	-0.0500	0.0358	0.0262	0.0290	0.0312	0.0251	0.0290	0.0315
30	-0.0460	0.0196	0.0219	0.0242	0.0261	0.0210	0.0242	0.0262
31	-0.0388	0.0219	0.0180	0.0199	0.0215	0.0173	0.0199	0.0216
32	-0.0385	0.0164	0.0204	0.0226	0.0243	0.0196	0.0226	0.0245
33	-0.0142	0.0358	0.0261	0.0289	0.0311	0.0250	0.0289	0.0313
34	0.0189	0.0276	0.0261	0.0289	0.0311	0.0250	0.0289	0.0313
35	0.0533	0.0246	0.0223	0.0247	0.0266	0.0214	0.0247	0.0268
36	0.0777	0.0220	0.0208	0.0230	0.0247	0.0199	0.0230	0.0249
37	0.0805	0.0232	0.0194	0.0215	0.0231	0.0186	0.0215	0.0233
38	0.0726	0.0175	0.0189	0.0210	0.0225	0.0181	0.0210	0.0227
39	0.0557	0.0256	0.0194	0.0215	0.0231	0.0186	0.0215	0.0233
40	0.0262	0.0171	0.0193	0.0214	0.0230	0.0185	0.0214	0.0232

POWER SPECTRUM ESTIMATION
FOR DISCRETE TIME SERIES
USING TUKEY-HANNING WEIGHTING FACTORS

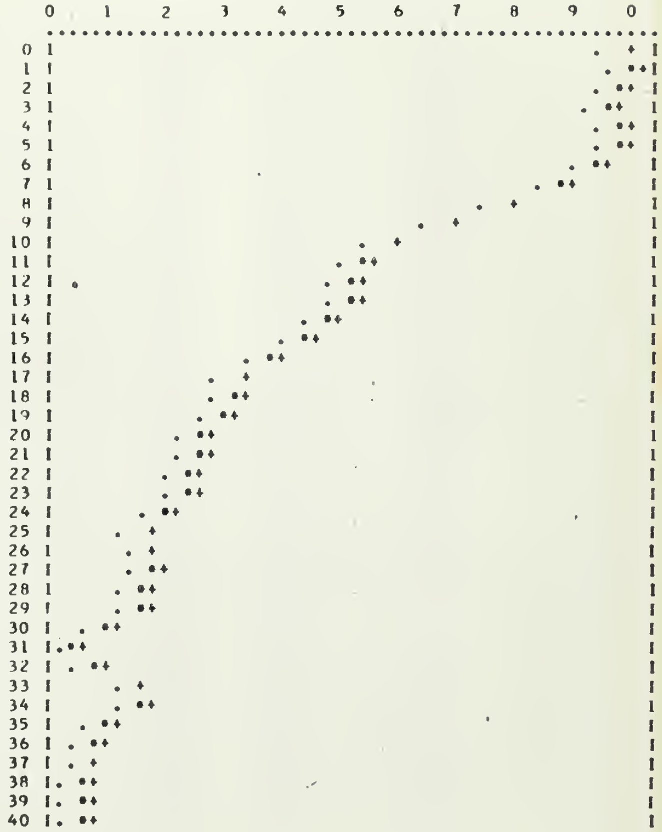
TWO SIZE BEADS   40 ROWS MIXED   LAG = 40

(2)

SMOOTHED POWER SPECTRUM
WITH 90 PCT. CONFIDENCE LIMITS



SMOOTHED POWER SPECTRUM
WITH 95 PCT. CONFIDENCE LIMITS



2 TIME 1.3033 MIN

POWER SPECTRUM ESTIMATION
FOR DISCRETE TIME SERIES
USING TUKEY-HANNING WEIGHTING FACTORS

THREE SIZE BEADS 40 ROWS MIXED LAG = 40

(3)

SMOOTHED SPECTRUM WITH INDICATED CONFIDENCE LIMITS

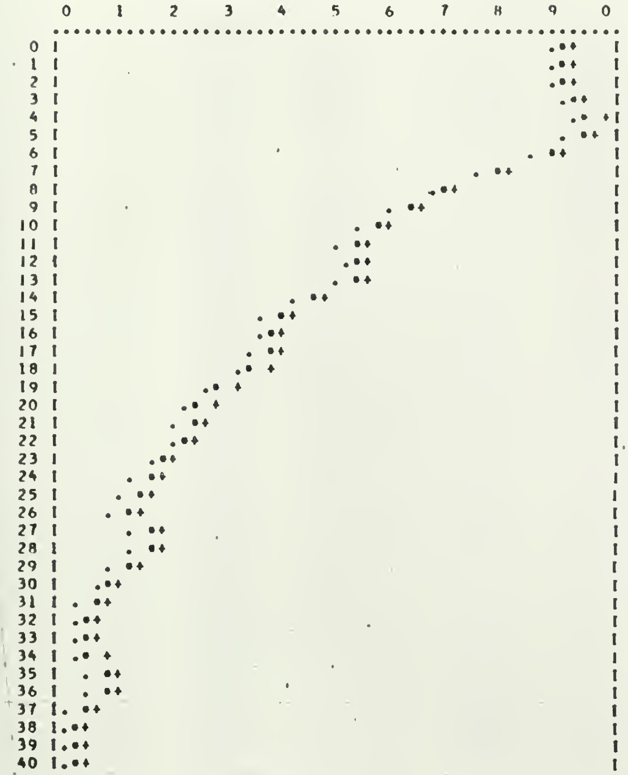
LAG	COVARIANCE SPECTRUM	RAW SPECTRUM	90 PCT. LOWER LIMIT	AVERAGE	90 PCT. UPPER LIMIT	95 PCT. LOWER LIMIT	AVERAGE	95 PCT. UPPER LIMIT
0	0.9035	0.4561	0.4371	0.4835	0.5203	0.4187	0.4835	0.5242
1	0.6015	0.5110	0.4342	0.4803	0.5168	0.4159	0.4803	0.5206
2	0.3676	0.4430	0.4367	0.4831	0.5198	0.4183	0.4831	0.5237
3	0.2062	0.5353	0.4675	0.5172	0.5565	0.4479	0.5172	0.5606
4	0.0977	0.5550	0.5005	0.5536	0.5957	0.4794	0.5536	0.6001
5	0.0241	0.5691	0.4802	0.5312	0.5715	0.4600	0.5312	0.5758
6	-0.0311	0.4313	0.3973	0.4394	0.4728	0.3806	0.4394	0.4764
7	-0.0683	0.3260	0.2918	0.3227	0.3473	0.2795	0.3227	0.3499
8	-0.0816	0.2076	0.2131	0.2357	0.2536	0.2041	0.2357	0.2555
9	-0.0742	0.2015	0.1719	0.1902	0.2046	0.1647	0.1902	0.2062
10	-0.0640	0.1502	0.1417	0.1568	0.1687	0.1358	0.1568	0.1699
11	-0.0298	0.1251	0.1219	0.1348	0.1451	0.1168	0.1348	0.1461
12	-0.0045	0.1388	0.1261	0.1395	0.1502	0.1208	0.1395	0.1513
13	0.0155	0.1554	0.1210	0.1338	0.1440	0.1159	0.1338	0.1451
14	0.0291	0.0857	0.0929	0.1028	0.1106	0.0890	0.1028	0.1114
15	0.0215	0.0844	0.0764	0.0845	0.0909	0.0732	0.0845	0.0916
16	0.0150	0.0835	0.0747	0.0826	0.0889	0.0715	0.0826	0.0896
17	0.0212	0.0791	0.0726	0.0803	0.0864	0.0696	0.0803	0.0871
18	0.0311	0.0797	0.0662	0.0732	0.0788	0.0634	0.0732	0.0794
19	0.0222	0.0546	0.0545	0.0603	0.0649	0.0523	0.0603	0.0654
20	0.0099	0.0526	0.0477	0.0528	0.0568	0.0457	0.0528	0.0572
21	-0.0087	0.0513	0.0465	0.0515	0.0554	0.0446	0.0515	0.0558
22	-0.0122	0.0508	0.0440	0.0487	0.0524	0.0422	0.0487	0.0528
23	-0.0044	0.0420	0.0386	0.0427	0.0459	0.0370	0.0427	0.0463
24	0.0009	0.0361	0.0350	0.0387	0.0416	0.0335	0.0387	0.0419
25	0.0088	0.0406	0.0326	0.0361	0.0388	0.0312	0.0361	0.0391
26	0.0103	0.0271	0.0315	0.0348	0.0375	0.0302	0.0348	0.0378
27	0.0014	0.0446	0.0353	0.0390	0.0420	0.0338	0.0390	0.0423
28	-0.0139	0.0397	0.0354	0.0391	0.0421	0.0339	0.0391	0.0424
29	-0.0078	0.0325	0.0309	0.0342	0.0368	0.0296	0.0342	0.0371
30	-0.0062	0.0320	0.0280	0.0310	0.0333	0.0268	0.0310	0.0336
31	-0.0009	0.0273	0.0256	0.0283	0.0304	0.0245	0.0283	0.0306
32	0.0053	0.0264	0.0246	0.0272	0.0293	0.0236	0.0272	0.0295
33	-0.0058	0.0287	0.0243	0.0269	0.0289	0.0233	0.0269	0.0291
34	-0.0093	0.0237	0.0247	0.0273	0.0294	0.0236	0.0273	0.0296
35	-0.0027	0.0331	0.0273	0.0302	0.0325	0.0262	0.0302	0.0327
36	-0.0024	0.0310	0.0272	0.0301	0.0324	0.0261	0.0301	0.0326
37	-0.0123	0.0253	0.0240	0.0266	0.0286	0.0230	0.0266	0.0288
38	-0.0208	0.0247	0.0228	0.0252	0.0271	0.0218	0.0252	0.0273
39	-0.0281	0.0261	0.0228	0.0253	0.0272	0.0219	0.0253	0.0274
40	-0.0367	0.0241	0.0227	0.0251	0.0270	0.0217	0.0251	0.0272

POWER SPECTRUM ESTIMATION
FOR DISCRETE TIME SERIES
USING TUKEY-HANNING WEIGHTING FACTORS

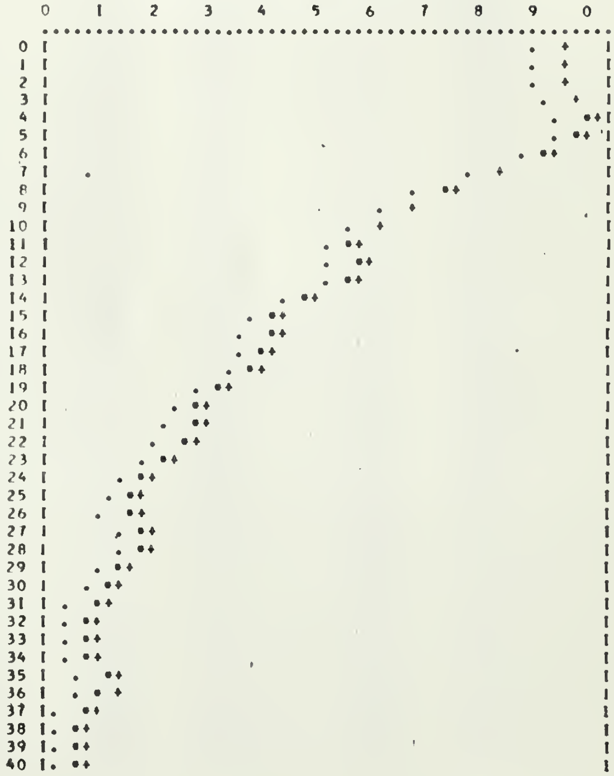
THREE SIZE READS 40 ROWS MIXED LAG = 40

(3)

SMOOTHED POWER SPECTRUM
WITH 90 PCT. CONFIDENCE LIMITS



SMOOTHED POWER SPECTRUM
WITH 95 PCT. CONFIDENCE LIMITS



2 TIME 1.3069 MIN

POWER SPECTRUM ESTIMATION
FOR DISCRETE TIME SERIES
USING TUKEY-HANNING WEIGHTING FACTORS

THREE SIZE BEADS   40 ROWS MIXED   LAG = 40

(4)

SMOOTHED SPECTRUM WITH INDICATED CONFIDENCE LIMITS

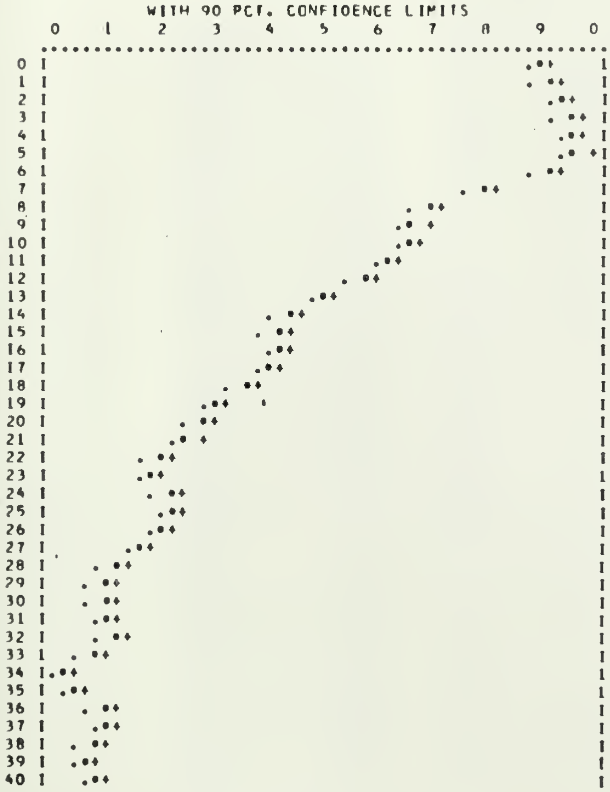
LAG	COVARIANCE SPECTRUM	RAW SPECTRUM	90 PCT. LOWER LIMIT	AVERAGE	90 PCT. UPPER LIMIT	95 PCT. LOWER LIMIT	AVERAGE	95 PCT. UPPER LIMIT
0	0.9281	0.4511	0.4243	0.4694	0.5050	0.4065	0.4694	0.5088
1	0.6461	0.4876	0.4451	0.4924	0.5298	0.4264	0.4924	0.5338
2	0.4014	0.5434	0.4865	0.5382	0.5791	0.4660	0.5382	0.5834
3	0.2204	0.5783	0.5042	0.5578	0.6002	0.4830	0.5578	0.6046
4	0.0973	0.5311	0.5217	0.5771	0.6209	0.4997	0.5771	0.6255
5	0.0130	0.6677	0.5312	0.5876	0.6323	0.5089	0.5876	0.6370
6	-0.0351	0.4839	0.4350	0.4812	0.5178	0.4168	0.4812	0.5217
7	-0.0712	0.2895	0.2874	0.3179	0.3421	0.2753	0.3179	0.3446
8	-0.0863	0.2087	0.2024	0.2239	0.2409	0.1939	0.2239	0.2427
9	-0.0818	0.1884	0.1824	0.2018	0.2171	0.1748	0.2018	0.2188
10	-0.0773	0.2216	0.1781	0.1970	0.2119	0.1706	0.1970	0.2135
11	-0.0682	0.1562	0.1552	0.1717	0.1848	0.1487	0.1717	0.1861
12	-0.0341	0.1528	0.1282	0.1418	0.1526	0.1228	0.1418	0.1537
13	0.0101	0.1055	0.1005	0.1111	0.1196	0.0962	0.1111	0.1205
14	0.0305	0.0808	0.0780	0.0863	0.0929	0.0748	0.0863	0.0936
15	0.0496	0.0783	0.0739	0.0818	0.0880	0.0708	0.0818	0.0887
16	0.0374	0.0898	0.0754	0.0834	0.0897	0.0722	0.0834	0.0904
17	0.0146	0.0756	0.0701	0.0776	0.0835	0.0672	0.0776	0.0841
18	0.0006	0.0693	0.0586	0.0648	0.0697	0.0561	0.0648	0.0703
19	-0.0117	0.0450	0.0484	0.0536	0.0577	0.0464	0.0536	0.0581
20	-0.0219	0.0550	0.0445	0.0492	0.0529	0.0426	0.0492	0.0533
21	-0.0409	0.0418	0.0403	0.0446	0.0480	0.0386	0.0446	0.0483
22	-0.0361	0.0398	0.0336	0.0372	0.0400	0.0322	0.0372	0.0403
23	-0.0300	0.0273	0.0317	0.0350	0.0377	0.0303	0.0350	0.0380
24	-0.0171	0.0458	0.0359	0.0397	0.0427	0.0344	0.0397	0.0430
25	-0.0093	0.0399	0.0375	0.0415	0.0446	0.0359	0.0415	0.0449
26	-0.0001	0.0402	0.0347	0.0384	0.0413	0.0333	0.0384	0.0416
27	0.0065	0.0333	0.0300	0.0332	0.0357	0.0287	0.0332	0.0360
28	0.0055	0.0258	0.0247	0.0273	0.0294	0.0236	0.0273	0.0296
29	0.0096	0.0241	0.0230	0.0255	0.0274	0.0221	0.0255	0.0276
30	0.0225	0.0278	0.0231	0.0256	0.0275	0.0222	0.0256	0.0277
31	0.0228	0.0226	0.0238	0.0263	0.0283	0.0228	0.0263	0.0285
32	0.0294	0.0324	0.0249	0.0275	0.0296	0.0238	0.0275	0.0298
33	0.0209	0.0227	0.0214	0.0237	0.0255	0.0205	0.0237	0.0257
34	0.0073	0.0172	0.0177	0.0196	0.0211	0.0170	0.0196	0.0213
35	-0.0038	0.0214	0.0195	0.0216	0.0232	0.0187	0.0216	0.0234
36	-0.0111	0.0263	0.0231	0.0255	0.0275	0.0221	0.0255	0.0277
37	-0.0184	0.0280	0.0238	0.0264	0.0284	0.0228	0.0264	0.0286
38	-0.0232	0.0232	0.0214	0.0237	0.0255	0.0205	0.0237	0.0257
39	-0.0141	0.0204	0.0209	0.0232	0.0249	0.0201	0.0232	0.0251
40	0.0002	0.0287	0.0222	0.0246	0.0264	0.0213	0.0246	0.0266

POWER SPECTRUM ESTIMATION
FOR DISCRETE TIME SERIES
USING TUKEY-HANNING WEIGHTING FACTORS

THREE SIZE READS   40 ROWS MIXED   LAG = 40

(4)

SMOOTHED POWER SPECTRUM
WITH 90 PCT. CONFIDENCE LIMITS



SMOOTHED POWER SPECTRUM
WITH 95 PCT. CONFIDENCE LIMITS

2 TIME   1.3056 MIN

POWER SPECTRUM ESTIMATION
FOR DISCRETE TIME SERIES
USING TUKEY-HANNING WEIGHTING FACTORS

3MM BEADS 40 ROWS MIXED LAG = 40

(5)

SMOOTHED SPECTRUM WITH INDICATED CONFIDENCE LIMITS

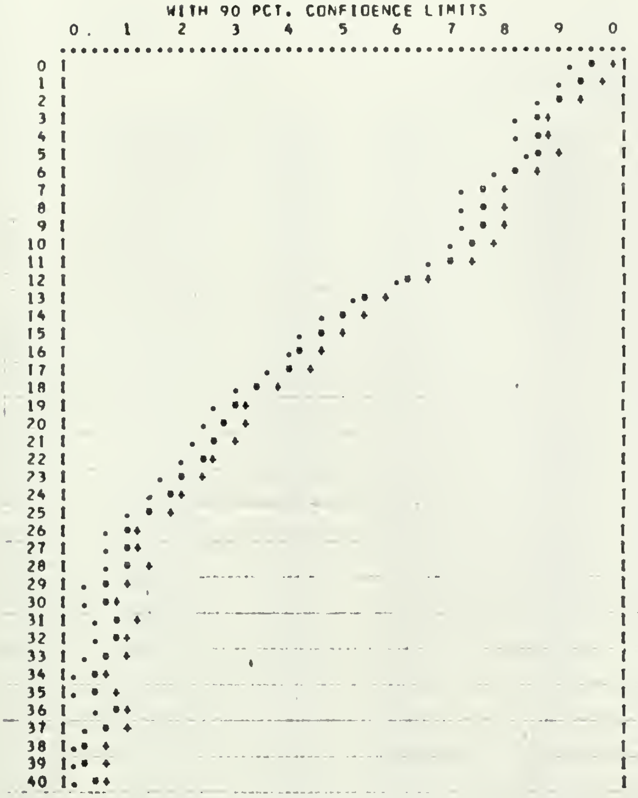
LAG	COVARIANCE SPECTRUM	RAW SPECTRUM	90 PCT. LOWER LIMIT	AVERAGE	90 PCT. UPPER LIMIT	95 PCT. LOWER LIMIT	AVERAGE	95 PCT. UPPER LIMIT
0	0.8960	0.4985	0.4128	0.4567	0.4914	0.3955	0.4567	0.4950
1	0.4939	0.4149	0.3748	0.4368	0.4700	0.3782	0.4368	0.4735
2	0.2325	0.4190	0.3541	0.3917	0.4215	0.3392	0.3917	0.4246
3	0.0761	0.3141	0.3093	0.3422	0.3682	0.2963	0.3422	0.3709
4	0.0184	0.3215	0.3090	0.3418	0.3678	0.2960	0.3418	0.3705
5	0.0049	0.4102	0.3251	0.3596	0.3869	0.3114	0.3596	0.3898
6	0.0114	0.2966	0.2851	0.3154	0.3394	0.2731	0.3154	0.3419
7	0.0276	0.2584	0.2473	0.2736	0.2944	0.2369	0.2736	0.2965
8	0.0287	0.2810	0.2475	0.2738	0.2946	0.2371	0.2738	0.2968
9	0.0223	0.2749	0.2460	0.2721	0.2928	0.2357	0.2721	0.2950
10	0.0125	0.2578	0.2307	0.2552	0.2746	0.2210	0.2552	0.2767
11	0.0036	0.2305	0.2071	0.2291	0.2465	0.1984	0.2291	0.2483
12	-0.0015	0.1976	0.1716	0.1898	0.2043	0.1644	0.1898	0.2058
13	0.0109	0.1337	0.1387	0.1534	0.1650	0.1328	0.1534	0.1663
14	0.0237	0.1485	0.1237	0.1368	0.1472	0.1185	0.1368	0.1483
15	0.0353	0.1166	0.1110	0.1228	0.1321	0.1063	0.1228	0.1331
16	0.0293	0.1093	0.1006	0.1113	0.1198	0.0964	0.1113	0.1207
17	0.0241	0.1101	0.0947	0.1048	0.1127	0.0907	0.1048	0.1136
18	0.0169	0.0896	0.0811	0.0897	0.0965	0.0776	0.0897	0.0972
19	0.0054	0.0694	0.0704	0.0778	0.0838	0.0674	0.0778	0.0844
20	0.0040	0.0830	0.0682	0.0755	0.0812	0.0654	0.0755	0.0818
21	-0.0150	0.0665	0.0646	0.0714	0.0769	0.0619	0.0714	0.0774
22	-0.0151	0.0697	0.0595	0.0658	0.0708	0.0570	0.0658	0.0714
23	-0.0165	0.0574	0.0550	0.0608	0.0655	0.0527	0.0608	0.0660
24	-0.0238	0.0589	0.0519	0.0574	0.0618	0.0497	0.0574	0.0623
25	-0.0298	0.0547	0.0475	0.0525	0.0565	0.0455	0.0525	0.0569
26	-0.0144	0.0418	0.0416	0.0460	0.0495	0.0399	0.0460	0.0499
27	-0.0053	0.0459	0.0415	0.0460	0.0494	0.0398	0.0460	0.0498
28	0.0126	0.0502	0.0424	0.0469	0.0504	0.0406	0.0469	0.0508
29	0.0179	0.0411	0.0379	0.0419	0.0451	0.0363	0.0419	0.0454
30	0.0056	0.0352	0.0369	0.0408	0.0439	0.0354	0.0408	0.0443
31	0.0147	0.0519	0.0402	0.0445	0.0478	0.0385	0.0445	0.0482
32	0.0188	0.0388	0.0397	0.0439	0.0472	0.0380	0.0439	0.0476
33	0.0254	0.0460	0.0379	0.0420	0.0451	0.0363	0.0420	0.0455
34	0.0202	0.0371	0.0357	0.0395	0.0425	0.0342	0.0395	0.0429
35	0.0142	0.0381	0.0359	0.0397	0.0427	0.0344	0.0397	0.0430
36	0.0119	0.0456	0.0397	0.0439	0.0472	0.0380	0.0439	0.0476
37	0.0025	0.0462	0.0386	0.0427	0.0459	0.0370	0.0427	0.0463
38	-0.0031	0.0327	0.0348	0.0385	0.0414	0.0333	0.0385	0.0417
39	0.0128	0.0421	0.0348	0.0385	0.0414	0.0333	0.0385	0.0417
40	0.0093	0.0370	0.0358	0.0396	0.0426	0.0343	0.0396	0.0429

POWER SPECTRUM ESTIMATION
FOR DISCRETE TIME SERIES
USING TUKEY-HANNING WEIGHTING FACTORS

3MM BEADS   40 ROWS MIXED   LAG = 40

(6)

SMOOTHED POWER SPECTRUM
WITH 90 PCT. CONFIDENCE LIMITS



SMOOTHED POWER SPECTRUM
WITH 95 PCT. CONFIDENCE LIMITS

2 TIME 1.3053 MIN

POWER SPECTRUM ESTIMATION
FOR DISCRETE TIME SERIES
USING TUKEY-HANNING WEIGHTING FACTORS

3MM BEADS 40 ROWS MIXED LAG = 40

(6)

SMOOTHED SPECTRUM WITH INDICATED CONFIDENCE LIMITS

LAG	COVARIANCE SPECTRUM	RAW SPECTRUM	90 PCT. LOWER LIMIT	AVERAGE	90 PCT. UPPER LIMIT	95 PCT. LOWER LIMIT	AVERAGE	95 PCT. UPPER LIMIT
0	0.9378	0.3632	0.3367	0.3725	0.4008	0.3225	0.3725	0.4037
1	0.5003	0.3817	0.3256	0.3602	0.3876	0.3119	0.3602	0.3904
2	0.2076	0.3141	0.3395	0.3755	0.4041	0.3252	0.3755	0.4071
3	0.0160	0.4923	0.3657	0.4045	0.4352	0.3503	0.4045	0.4385
4	-0.0592	0.3193	0.3115	0.3446	0.3708	0.2984	0.3446	0.3735
5	-0.0406	0.2473	0.2520	0.2788	0.2999	0.2414	0.2788	0.3022
6	0.0219	0.3011	0.2537	0.2807	0.3020	0.2431	0.2807	0.3042
7	0.0797	0.2732	0.2560	0.2832	0.3047	0.2453	0.2832	0.3070
8	0.0800	0.2854	0.2612	0.2889	0.3108	0.2502	0.2889	0.3132
9	0.0265	0.3115	0.2886	0.3193	0.3435	0.2765	0.3193	0.3461
10	-0.0284	0.3686	0.3366	0.3723	0.4006	0.3224	0.3723	0.4036
11	-0.0670	0.4406	0.3343	0.3698	0.3979	0.3202	0.3698	0.4009
12	-0.0634	0.2293	0.2334	0.2582	0.2778	0.2236	0.2582	0.2799
13	-0.0356	0.1336	0.1479	0.1636	0.1761	0.1417	0.1636	0.1774
14	-0.0014	0.1580	0.1246	0.1378	0.1483	0.1194	0.1378	0.1494
15	0.0178	0.1016	0.1045	0.1156	0.1244	0.1001	0.1156	0.1253
16	0.0163	0.1013	0.0926	0.1025	0.1103	0.0887	0.1025	0.1111
17	-0.0261	0.1057	0.0835	0.0923	0.0994	0.0800	0.0923	0.1001
18	-0.0426	0.0566	0.0692	0.0765	0.0824	0.0663	0.0765	0.0830
19	-0.0340	0.0872	0.0739	0.0817	0.0879	0.0708	0.0817	0.0886
20	-0.0078	0.0959	0.0744	0.0823	0.0886	0.0713	0.0823	0.0892
21	0.0252	0.0503	0.0660	0.0731	0.0786	0.0633	0.0731	0.0792
22	0.0457	0.0958	0.0636	0.0703	0.0757	0.0609	0.0703	0.0762
23	0.0485	0.0394	0.0544	0.0602	0.0648	0.0521	0.0602	0.0653
24	0.0337	0.0662	0.0529	0.0585	0.0629	0.0506	0.0585	0.0634
25	0.0168	0.0620	0.0552	0.0610	0.0657	0.0529	0.0610	0.0662
26	0.0096	0.0538	0.0481	0.0532	0.0573	0.0461	0.0532	0.0577
27	0.0069	0.0432	0.0431	0.0477	0.0513	0.0413	0.0477	0.0517
28	0.0445	0.0504	0.0426	0.0471	0.0507	0.0408	0.0471	0.0511
29	0.0595	0.0446	0.0429	0.0474	0.0510	0.0411	0.0474	0.0514
30	0.0164	0.0502	0.0458	0.0506	0.0545	0.0438	0.0506	0.0549
31	-0.0203	0.0575	0.0465	0.0514	0.0553	0.0445	0.0514	0.0557
32	-0.0226	0.0403	0.0415	0.0459	0.0494	0.0397	0.0459	0.0497
33	-0.0139	0.0453	0.0409	0.0452	0.0487	0.0392	0.0452	0.0490
34	-0.0204	0.0500	0.0425	0.0470	0.0506	0.0407	0.0470	0.0510
35	-0.0295	0.0429	0.0411	0.0455	0.0489	0.0394	0.0455	0.0493
36	-0.0031	0.0462	0.0386	0.0427	0.0459	0.0369	0.0427	0.0462
37	-0.0134	0.0354	0.0367	0.0405	0.0436	0.0351	0.0405	0.0440
38	-0.0295	0.0452	0.0369	0.0408	0.0439	0.0354	0.0408	0.0443
39	-0.0343	0.0375	0.0398	0.0441	0.0474	0.0382	0.0441	0.0478
40	-0.0150	0.0560	0.0423	0.0468	0.0503	0.0405	0.0468	0.0507

POWER SPECTRUM ESTIMATION
FOR DISCRETE-TIME SERIES
USING TUKEY-HANNING WEIGHTING FACTORS

3MM BEADS 40 ROWS MIXED LAG = 40

(6)